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ABSTRACT

This second booklet of a two-part publication, designed as an in-service tool for teachers of grades K-9, concerns the teaching and curriculum aspects of problem solving. It is a digest of selected topics integral to successful problem-solving instruction. The resource is organized into independent units for individual reference or group in-service use. These units can be employed in a variety of ways. Self-contained topics may be selected as needed by teachers engaging in problem-solving instruction with their classes. Individual units may serve as the topic for in-service meetings conducted by supervisory staff. Particular units can be selected for periodic sustained study throughout the school year as a comprehensive in-service program or course on problem-solving teaching. The individual unit titles are: Introduction to Teaching Problem Solving; Research and Psychological Summaries; Looking Back Strategies; Teaching Considerations; Choosing Different Teaching Modes; Story Problems and Number Sentences; Getting Started; Primary Instruction; Intermediate and Junior High Instruction; Creating a Problem Solving Environment; Organizing the Curriculum; and Calculators in Problem Solving. (Author/MK)

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PROBLEM SOLVING... a basic mathematics goal.

ED186288

A resource for problem solving

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FOREWORD

The ability to meet and overcome problems is a lifetime skill that every future citizen will need. Educating today's young people to meet the problems of tomorrow requires a careful blend of traditional basic skills instruction and opportunities to apply those skills to real situations. The person who is deficient in either the fundamentals or the techniques and strategies to apply them will be handicapped in his ability to cope with daily problems.

Problem solving has long been recognized as an educational goal. And as we move closer toward the 21st century, it is rapidly becoming an increasingly important area for competence. To equip students with the appropriate problem solving skills for future citizenship responsibilities will require careful curriculum planning, systematic instruction, and the commitment and professional expertise of teachers.

Problem Solving – A Basic Mathematics Goal has been prepared as an inservice and resource tool to assist teachers in improving their own problem solving skills and their teaching of those skills. This publication is a comprehensive source of content, methodology, and curriculum development information designed to make problem solving facility a more readily attainable goal.

Franklin B. Walter

Franklin B. Walter
Superintendent for Public Instruction

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INTRODUCTION TO THE PUBLICATION

Problem Solving – A Basic Mathematics Goal is a two-part publication designed as an inservice tool for teachers of grades K-9. Goals for this publication are: to help teachers become better mathematical problem solvers, to introduce techniques and materials helpful in improving student problem solving skills, and to suggest ways to organize the curriculum and daily instruction to achieve problem solving goals.

Becoming A Better Problem Solver is the introductory booklet of the publication. It is based upon the postulate that a teacher should become a better problem solver in order to teach problem solving more effectively. The booklet provides an overview of problem solving as a process used daily by all individuals and as a curriculum goal important to all disciplines. A substantial portion of the booklet is devoted to introducing ***strategies***, methods useful in getting started on problems and successfully solving them. Though the booklet uses mathematical problems as the medium for problem solving discussions and activities, the techniques have application to virtually any topic.

The second booklet of this publication concerns the teaching and curriculum aspects of problem solving. **A Resource for Problem Solving** is a digest of selected topics integral to successful problem solving instruction. Arranged in a topical format of associated ideas, sections are written for independent teacher use. These sections may be studied sequentially or referred to singly, as needed.

This two-part publication is written in recognition of and commitment to problem solving as a fundamental goal of education within reach of every student. Though problem solving skills are generally recognized as important attributes of the educated person, past development of these skills has been more the product of chance and individual effort than well-defined curriculum planning. This publication is an attempt to provide definition and direction to teaching efforts to achieve problem solving outcomes in one curriculum area – mathematics – but with obvious bridges to other subjects.

ABOUT THIS RESOURCE

This resource is organized into independent units for individual reference or group inservice use. These units can be employed in a variety of ways. Self-contained topics may be selected as needed by teachers engaging in problem solving instruction with their classes. Individual units may serve as the topic for inservice meetings conducted by supervisory staff. Particular units can be selected for periodic sustained study throughout the school year as a comprehensive inservice program or course on problem solving teaching.

Persons wishing to use the resource for more lengthy inservice programs are advised to conduct a preliminary needs assessment meeting to ascertain staff interests and needs. Selection and organization of units should then follow accordingly. To facilitate group or individual use of this resource, a brief description of individual units follows.

Introduction to Teaching Problem Solving provides an overview of the task of teaching problem solving and offers a rationale for considering problem solving as a basic skill which is the responsibility of every teacher to help develop. *Research and Psychological Summaries* is a short review of research findings and psychological views about factors that influence problem solving development.

Five additional strategies are introduced in *Looking Back Strategies.* These strategies are considered more appropriate for teaching use in large group instruction than for initial, individual use by students. *Teaching Considerations* is a compendium of suggestions ranging from teaching skills to the teaching role in large or small group instruction.

Choosing Different Teaching Modes addresses the need to provide variety in the way that problems are presented to students. It examines problem solving as a real-life activity which involves meeting problems in many different forms. *Story Problems and Number Sentences* discusses some of the complexities in teaching young children how to understand story problems and to write related number sentences.

A number of helpful ideas for getting a problem solving strand underway are offered in *Getting Started.* Suggestions begin with making better use of standard textbook materials and expand to generating good problems from other resources. *Primary Instruction* lists K-3 goals for problem solving instruction and provides examples of primary level strategies and problems. A companion unit for grades 4-9 is *Intermediate and Junior High Instruction.*

The importance of organizing the learning area is the subject of *Creating a Problem Solving Environment.* Suggestions for providing an appropriate climate and instructional resource area are listed. *Organizing the Curriculum* discusses considerations for planning a K-10 problem solving strand in mathematics. This unit concludes with a scope and sequence chart of objectives, including affective learning characteristics.

Calculators in Problem Solving explores the potential of the hand calculator for problem solving use. Included are guidelines for determining when the use of a calculator is appropriate or inappropriate. The resource concludes with a list of 75 grade-referenced problems that can serve as the nucleus for an individual classroom problem solving file. Additional resources for good problems and for problem solving materials are included as selected references in the bibilography section.

Throughout **A Resource for Problem Solving**, example problems are used to illustrate points about problem solving. Each of these problems carries a suggested grade-level designation. This is a general reference which may or may not be appropriate for a given class depending upon students' previous problem solving experiences. Teachers selecting these materials should use their own discretion in adapting problems for a particular class of students.

The principal purpose for **A Resource for Problem Solving** is to help teachers achieve their goals for improving student problem solving capability. It is a resource that a teacher may decide to return to many times as different units are required. Topics represent problem solving in the broadest sense – instructional goals, methodology, and teaching philosophy – in addition to the specific techniques for solving problems. As such, the suggestions and comments of this resource can be appropriately applied to many other areas of mathematics instruction as well as to other content disciplines.

INTRODUCTION TO TEACHING PROBLEM SOLVING

A Problem Solving Conversation

Dad: "What's the matter son?"

Boy: "Oh . . . I have this homework to do on fractions, and I don't know how to do it."

Dad: "Been having any problems before this?"

Boy: "No, it's just that I missed three days of school when these problems were discussed. And now I have to do this makeup work."

Dad: "What's one of your problems?"

Boy: "Well, the first question is $4\frac{2}{3} \times 3\frac{4}{5}$."

Dad: "Do you know how to do anything similar to this problem?"

Boy: "Let me think . . . I know how to multiply fractions! $\frac{2}{3} \times \frac{4}{5}$ is just $\frac{2 \times 4}{3 \times 5} = \frac{8}{15}$."

Dad: "Are only fractions involved?"

Boy: "No, there are whole numbers, too."

Dad: "Whole numbers and fractions . . . "

Boy: "Yeah . . . We studied multiplying fractions by whole numbers.

Dad: "How do you do that?"

Boy: "You just write the whole number like a fraction with a denominator of 1 and then multiply like fractions.

Dad: "Such as . . . "

Boy: "If you had $6 \times \frac{2}{5}$, you write it as $\frac{6}{1} \times \frac{2}{5} = \frac{6 \times 2}{1 \times 5} = \frac{12}{5}$ which you change to $2\frac{2}{5}$."

Dad: "But this is not getting your problem solved."

Boy: "Come on, Dad. Can you show me how to do it. I've got to have this by tomorrow."

Dad: "Well, I can't show you how to do your homework. But maybe I can ask you some questions that can help you to come up with the right answers."

Dad: "Suppose that you are on a deserted island, and the only way that you can get off is to answer this question correctly."

Boy: "That's corny, Dad."

Dad: Yes, I know. But sometimes 'corny' situations help you to think better. For example, try to think of an interesting situation that involves multiplying, $4\frac{2}{3}$ by $3\frac{4}{5}$."

Boy: "Okay, you asked for it. That deserted island that I'm on isn't really deserted at all. There are four men there with me and we want to send a message to the mainland for pizzas. Each man wants $3\frac{4}{5}$ pizzas. But I can't eat as much as they can. So suppose I can eat $\frac{2}{3}$ of that amount. We have to determine how many pizzas to ask for in the note in the bottle."

Dad: "What note in the bottle?"

Boy: "Well, it's a deserted island, right. That means no phones or anything else. So we have to order our pizzas by throwing a bottle in the ocean."

Dad: "Okay . . . your corny situation will work for our purposes. Now think about the men only. What do you have to do to order for them?"

Boy: "I don't see what you mean."

Dad: "How many whole pizzas does each man eat, and what fractional part does each man eat?"

Boy: "Oh, I get it. Each man would eat 3 whole pizzas. So that is 4 x 3 = 12 whole pizzas. Each one also eats $\frac{4}{5}$ of another pizza, so that makes $4 \times \frac{4}{5} = \frac{4}{1} \times \frac{4}{5} = \frac{16}{5}$ or $3\frac{1}{5}$ more pizzas."

Dad: "Very good . . . so far."

Boy: "Then, I have to consider me. Since I eat $\frac{2}{3}$ as much as one of them, I want $\frac{2}{3} \times 3$ whole pizzas. That's just 2 pizzas. And I also have to order $\frac{2}{3} \times \frac{4}{5}$ of a pizza, which is $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$ of a pizza"

Dad: "How many pizzas is that altogether?"

Boy: "Wait a minute . . . I'm getting there."

$12 + 3\frac{1}{5} + 2 + \frac{8}{15} = 17 + \frac{3}{15} + \frac{8}{15} = 17\frac{11}{15}$ pizzas that we have to order."

Dad: "So that is what you would write on the note?"

Boy: "No, if they were really pizzas, we would have to order 18."

Dad: "Very good. Now, let's look back, and examine what you did with those mixed numbers."

* * *

The conversation above between a sixth grade boy and his father is unusual in many respects. The warm and comfortable teaching-learning rapport that exists between the two apparently has been developed long before this incident. The boy is alert and capable, and has been snagged on a very small problem. But he quite obviously has a very thorough grasp of previous work leading up to this problem. The father is unusually knowledgeable and insightful about the learning task confronting his son. He participates in this learning situation in such a way that the thinking, discovery, and solution of the problem are accomplished by the boy.

Perhaps most unusual of all is the amount of time that is devoted to learning how to solve this computational exercise. The boy and the father are apparently motivated by some other goal, some more important outcome, than learning how to multiply two mixed numbers. That goal is problem solving, and the father is our means to introduce the role that teaching plays in problem solving learning.

As we analyze the conversation, several of the father's actions begin to stand out. Early in the conversation, he asks the question, "Been having any problems before this?", seeking to establish background information. He wants to *determine the relationship of this difficulty to other learning skills* that may be involved. He *asks the boy to outline the problem* and then *guides him to identify needed relationships through careful questioning.*

Early in the problem, he leads the boy into *reconsidering the simpler related problems* of multiplying fractions by whole numbers and other fractions.

In asking his son to place the problem in a deserted island situation, he accomplishes three things: (1) he *eases any possible tension* in the situation by making it humorous and desireable to proceed further; (2) he causes the boy to *personalize and identify with the problem;* and (3) he *casts the problem in a realistic setting* in which the boy can better visualize the numbers in terms of objects he can conceptualize. All of these actions enhance the learning situation from a problem solving perspective.

When the boy gets stuck, the father *offers another hint, but not so explicit that opportunity for discovery is lost.* Later, with the question, "So that is what you would write on the note?", we see a step so often missing in the solution of a problem. *Does the answer make sense?* Is it practical? How accurate does the answer have to be? With conversation drawing to a close, a very important looking back strategy is hinted at by the father. "Now, let's look back, and examine what you did with those mixed numbers."

In this conversation, we have seen modeled by the son and father several of the important components of a teaching-learning situation that involve problem solving instruction. This resource unit deals with these and other important teaching and curriculum considerations which you the teacher can use to advantage in improving your students' problem solving learning.

Problem Solving Is A Basic Skill

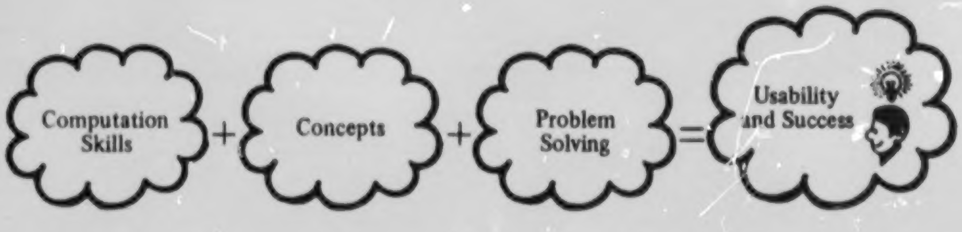
> "I think problem solving is the basic skill in mathematics. By problem solving I mean more than knowing what to do, in the sense of having knowledge. I mean having a kind of commitment to problem solving – a willingness to tackle a problem even when one doesn't know right away what to do, and to keep plugging away at the problem until one finds a reasonable solution." (Georgia state legislator, 1977)

Computational proficiency and numerical concept development have long been recognized as basic skill goals of elementary and middle school programs. Though mentioned in most curriculum descriptions, problem solving has seldom been pursued with the same immediacy or emphasis as these other two goals. Characteristically, problem solving has been perceived as a by-product of other learning activities that are considered to be of greater value.

However, recent interest in the effectiveness of school programs is causing reevaluation of some program goals. Assessment results are highlighting an interesting fact – computation skill proficiency is not indicative of the ability to apply such skills to meaningful situations. Consider, for example, the following Ohio assessment results (see following page).

Marked discrepancies between performances on computation items and application items are cause for reappraising what we mean by basics. Certainly, computational proficiency and understanding of number concepts should continue to represent important mathematics goals for universal mastery. But their mastery alone is not sufficient to provide a basic education. We must also be concerned with what students can do with those skills – with how well they can apply such skills and ideas to situations they will face as future citizens.

When confronting a problem situation, learners have to draw upon their knowledge for three things: (1) empirical skills (e.g. reading, computation) that permit them to interact with problem information – reshaping, reorganizing, and extending; (2) concepts that allow them to understand and operate within the problem situation; and (3) problem solving skills that enable them to direct their personal resources to remove any blockage preventing solution of the problem. Whenever any of these three requisites is missing or weak, learners may not possess sufficient skills to resolve problem situations necessary to satisfy their needs.



Traditionally, problem solving has received only sporadic curriculum treatment, and this from teachers concerned with how other skills are to be applied. But we can no longer afford incidental development of problem solving abilities. If students are to develop the educational skills required to cope with the growing citizenship responsibilities of the future, problem solving skills will have to be elevated to the same level of priority and emphasis as other basics.

Items from Ohio Eighth-Grade Mathematics Assessment (1977)

(Correct answers and percent of students choosing that response are indicated)

$$\begin{array}{r} 373 \\ -\ 194 \\ \hline \end{array}$$

*A. 179 (94%)
B. 189
C. 221
D. 289

E. I don't know.

$$\begin{array}{r} 64 \\ \times\ 35 \\ \hline \end{array}$$

A. 99
B. 512
C. 1,240
*D. 2,240 (90%)

E. I don't know.

$\frac{4}{5} - \frac{1}{2} =$ _____

A. $1\frac{1}{3}$

B. $\frac{5}{7}$

C. $\frac{3}{7}$

*D. $\frac{3}{10}$ (67%)

E. I don't know.

0.941 - 0.36 = _____

*A. 0.581 (63%)
B. 0.905
C. 1.301
D. 5.81

E. I don't know.

If John drives his car at an average speed of 50 miles per hour, how long will it take him to drive 275 miles?

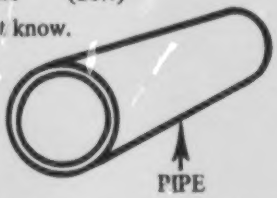
A. 5 hours, 5 minutes.
B. 5 hours, 25 minutes.
*C. 5 hours, 30 minutes (32%)
D. 2 hours, 5 minutes

E. I don't know.

The radius of the inside of the pipe is 3 inches. The wall of the pipe is 1 inch thick. What is the diameter of the pipe?

A. 4 inches
B. 5 inches
C. 7 inches
*D. 8 inches (20%)

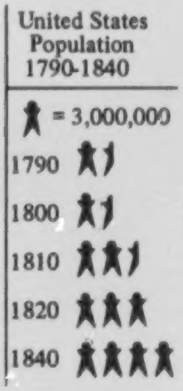
E. I don't know.



What was the approximate population in the United States in 1810?

A. 1,500,000
B. 2,500,000
C. 6,500,000
*D. 7,500,000 (35%)

E. I don't know.



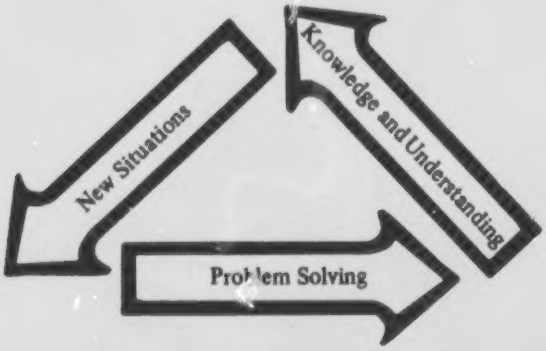
Results on whole number, fraction, and decimal computations averaged 81.5%.
Results on written problem applications of these operations averaged 57.9%.

Your Responsibility

Problem solving is perceived by some educators as a higher intellectual activity that should be deferred until fundamentaal concepts and computation skills are well in place. Believing their responsibilities to involve the development of more fundamental skills, they leave problem solving instruction for others at a later time.

But problem solving is a natural learning activity that every preschool child has already experienced. It begins as soon as an infant becomes aware of himself and his environment, develops some need, and tries to satisfy it. He soon discovers a cause and effect relationship between his actions and the satisfaction of his immediate needs. Some actions produce satisfactory results. Some do not. However, each experience with attempting to solve a problem adds to the infant's understanding and growth.

As the infant develops into a young child, he uses additional information, capabilities, and knowledge to expand his range of responses to problems. Utilizing his growing skills, the young learner is able to explore new situations that generate still further problems for mastery. In the process of conquering these new problems, the information and experience gained through each solution become part of his growing problem solving capability, subtly reshaping his existing knowledge and skills through its acquisition.



A good education program builds on the natural, informal methods the child brings to school. It provides a balance of skill, concept, and problem solving development similar to the kind of learning the child has experienced prior to coming to school. No aspect of learning is sacrificed, postponed, or neglected. All requisites for the application of learning are pursued concurrently, each reinforcing and strengthening the other. Skills and concepts are applied in problem solving, and problem solving provides a means to practice skills and strengthen content understandings.

Problem solving thereby becomes the responsibility of every teacher K-12. To ascribe a prerequisite status to problem solving is to interrupt the natural mode of learning and to run the risk that some children will never have the formal opportunity to participate in problem solving development because they have not achieved other goals.

Teaching Priority

The role and importance that a teacher assigns to problem solving will directly affect the growth of his students. If he does not place the same value on problem solving objectives that he does on other instructional goals, neither will his students. But if instructional time and evaluation procedures make problem solving a significant part of the program, students will respond accordingly.

We have suggested that problem solving should stress the thinking processes involved in reaching a solution. Overemphasis on answers, computation skills, or procedures will detract from student appreciation of the thinking skills required for solving problems. Consider, for example, the continuing discussion of the sixth-grade boy with his father.

Boy: "Hey, Dad. You know that problem you helped me with the other day."

Dad: "What's that? Oh, ... you mean the mixed number problem."

Boy: "Yeah, that's the one. Well, in school, we aren't doing those problems the way I discovered when I was talking with you."

Dad: "That's not surprising."

Boy: "When we take that problem, $4\frac{2}{3} \times 3\frac{4}{5}$, we convert the mixed numbers to improper fractions first, and multiply, $\frac{14}{3} \times \frac{19}{5}$."

Dad: "Oh . . . do you get the same answer?"

Boy: "Sure, but you wind up multiplying and dividing some real large numbers. Look -- $\frac{14}{3} \times \frac{19}{5}$ becomes $\frac{14 \times 19}{3 \times 5} = \frac{266}{15}$. Then you have to divide by 15!"

Dad: "You eventually get the same answer, what was it . . . $17\frac{11}{15}$?"

Boy: "Yeah, but that's not the point. My way gets the same answer and keeps the numbers smaller."

Dad: "What does your teacher say about that?"

Boy: "Well, Ms. Williams was really interested in my way of doing the problem. And after I worked some problems out of the book using the book's way, I talked with her again."

Dad: "And . . ."

Boy: "I explained how I thought my way was better and asked if I could use it. She thought about it for a minute and said that I could use my method after I bring her an example tomorrow where the book technique is shorter."

Fortunately, this teacher appreciated the problem solving considerations that were present in the learning situation. Imagine how the boy's enthusiasm would have been cooled by insistence on using only one procedure. He might approach the next opportunity for discovery and personal invention of mathematics differently if his ideas had been dismissed lightly. The teacher commended the boy and showed interest in his discovery. When the situation warranted, she took advantage of the opportunity to use her student's enthusiasm to have him look into the situation further.

This teacher showed an excellent understanding of the need for balance in achieving learning objectives. Computation and problem solving outcomes were involved in the learning situation. Neither was sacrificed for the other.

A Common Goal

Problem solving is more than obtaining answers. It entails more than thinking skills and problem-reducing strategies. It is partly cognitive, partly affective. It is a tool, a means of thinking, and a philosophy. It is a predisposition to learn from every available opportunity the most that can be gleaned from that experience.

In teaching problem solving, our goal is for students to acquire an appreciation for the comprehensive nature of problem solving as well as to develop specific skills. It is an ambitious goal but a shared responsibility. Its attainment is the result of many years of good instruction, well-planned experiences, and individual student involvement.

The following analogy comparing problem solving to mountain climbing can be useful in providing insight into the many facets of problem solving.

Attempting the scaling of a mountain peak is not a task to be undertaken lightly nor without adequate preparation. The climber must want to accomplish the feat, recognize that there will be many obstacles and possible setbacks, and admit the possibility that his effort may not be successful. However, he is confident that his abilities are sufficient to the task. These abilities have been developed over time and tested before, though perhaps, in less trying circumstances. His individual skills are many and varied. But seldom have all been called into play for a single situation. There is no prescribed sequence in which his skills are employed. However, experience has shown that some combinations are more effective in handling certain situations.

Many kinds of tools and instruments are available for the good climber to draw upon to help him with his task. The more difficult the climb, the more sophisticated are the pieces of helping apparatus that he may require. Before the climb is attempted, he gives some attention to selecting the kinds of equipment and material that he may need to employ.

The scaling of the mountain actually begins with the climber's careful study of the peculiarities of his task, the obstacles he must overcome, the resources and options he has available, the goals he desires. Although his final goal may be to achieve a certain spot, the routes that he may take may be many and varied, representing distinct challenges of differing degrees of difficulty if the peak is attempted by that route.

Selecting a course of action, the climber begins. At the early stages, he may have short-range, immediate objectives that are part of an overall plan. But his plan is flexible. Details are to be determined by the climber only as the climb itself proceeds. New obstacles are expected to occur during the course of the ascent, each requiring his skill and ingenuity to surmount.

Occasionally, the climber may find himself in a situation where an obstacle is too great to be overcome directly. He may have to approach it from another direction, possibly backtracking until a more productive means of overcoming the obstacle is apparent. The obstacle might even prove so awesome that the climber retreats to the base of the mountain and begins anew.

In this event, the climber may appear to be no nearer to realizing his goal than when he first began. However, he has gained useful information about the nature of the peak, its problems and difficulties. Aware now of a path that is not productive is useful information to him. Now alerted to a healthy respect for this particular mountain, he may bring additional resources to bear in attempting to master it. The climber is not necessarily daunted by his present failure to reach his goal because he has anticipated a temporary setback as one possibility. Instead, he renews his efforts with even more commitment and zeal.

Eventually, the climber masters the peak (or realizes that it is presently beyond his skill). Pride and sense of accomplishment are his. The process of employing his individual skills and resources to overcome this challenge produces satisfaction and excitement. His feelings of success have little relationship to how others view this particular peak or the fact that it may have been scaled by many others before. The challenge of conquering a peak is a very individual kind of experience that is measured in individual terms – the skills of the climber against the challenges of the peak.

But the climb is not yet complete even though the exhiliration is felt at this point. The climb must be described to others, the results recorded, the experience finalized. The climber must descend.

After the climber is back on the ground, he may reflect upon what he has achieved. Appreciation of his present accomplishment may extend to recognizing the significance of his present actions to peaks yet to be mastered. Certainly, he must enjoy far greater insight about the specific obstacles his efforts have recently overcome. He may have grown considerably in his ability to assess the difficulty of specific obstacles associated with the climb. He may have greater insight into ways that he will approach the difficulties of future challenges. He may very well conclude that if he were to start anew on this peak that he would approach it in a new and, hopefully, more satisfying way.

The true mountain climber also recognizes that this particular mountain warrants even further consideration. He has mastered it from only one perspective. As a result he understands the mountain much better. But he recognizes that the mountain may yield even further secrets and knowledge about itself as well as the process of mountain climbing through another effort from a totally different approach.

So it happens with problem solving in general. Problem solving involves the utilization of subordinate skills, involves attitudes about the process and oneself, and involves many stages. Even setbacks are advances in knowledge about the problem. Problems are seldom solved by even, steady progress toward the goal. In the early stages, the solver may depend upon intuition as well as highly structured planning. In seeking to resolve the problem, the solver may construct subgoals to be achieved as part of the overall task.

At some point, the last major obstacle will be overcome, and the solver will enjoy the exhilaration of the 'eureka experience.' But the problem is not complete until he has transformed this result into meaningful terms that can be communicated. Even then, he may wish to extend the experience by looking back at the problem and considering what he has learned about himself, the process, and the nature of the problem in solving it. The answer is only a small part of that overall experience.

RESEARCH AND PSYCHOLOGICAL SUMMARIES

A Research Self-Quiz

Test your knowledge of problem solving in the elementary grades by predicting the answers to these true-false statements. Compare your answers to the summaries of findings.

Children are more successful at solving problems when:

- *problem settings are similar to their environments.* (a) T F
- *problems involve subtraction than addition.* (b) T F
- *problem information is presented in the order in which it will be used.* (c) T F
- *students work together in small groups.* (d) T F

(a) True, but only slightly. Children have more success with problems in which settings are familiar, but problems with unfamiliar settings do not seem to cause undue difficulty. Findings generally show that children enjoy a variety of problem settings and that youngsters need to be interested in the problems they are asked to solve.

(b) False. Problems involving addition are judged the easiest – followed in difficulty by subtraction, multiplication, and division problems. Problems that involve only one operation are usually easier than multistep problems requiring more than one of the basic operations.

(c) True. However, it must pointed out that data in real-life situations seldom come in convenient sequencing.

(d) True. Working together in small groups, students can solve more problems than they can working individually. Moreover, they seem to benefit from the discussion required to achieve solutions. The quality of the solutions may or may not be better than the independent solutions of the most capable group member, depending upon on whether he or she is identified in that role by the group.

Problem solving abilities of children are enhanced by:

- *emphasizing word cues as clues to solving problems.* (e) T F
- *having students write open sentences for problems.* (f) T F
- *encouraging variety in the ways that problems are solved.* (g) T F
- *permitting children to use calculators to solve problems.* (h) T F

(e) False. Associating specific problem actions with cue words can be misleading and counterproductive. Attention is directed away from the identification of problem relationships that are instrumental in achieving solution and perhaps crucial to understanding what is involved. However, specific instruction on quantitative vocabulary can be helpful for some students.

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*Adapted with permission from Suydam, Marilyn N. and Weaver, J. Fred, "Research on Problem Solving: Implications for Elementary School Classrooms." **The Arithmetic Teacher** (Nov 1977): 40–42.

(f) True. Having students write open sentences (where they are appropriate) has two advantages: to represent actions or relationships of the problem, and to emphasize operations that must be used to solve the problem.

(g) True. Encouraging a child to use a variety of means to solve the same or similar problems expands his problem solving abilities as does the opportunity to see other children's solution techniques.

(h) The research jury is still out on this question. However, preliminary reports by leading mathematics educators suggest that calculators *when used in problem solving settings* have great potential for extending the range of problems to be considered, for freeing a child's thinking from the computation aspects of a problem, and for motivating a child to place more emphasis on the relationships within a problem.

Better problem solvers usually:

• *are girls in the primary grades and boys in the intermediate grades.*	(i)	T	F
• *have higher IQ's.*	(j)	T	F
• *come from higher socioeconomic groups.*	(k)	T	F
• *are better readers.*	(l)	T	F

(i) False. There appear to be no sex differences in abilities to solve problems.

(j) True. IQ scores are significantly related to problem solving ability.

(k) False. As a single factor, socioeconomic standing does not appear to be significant in problem solving ability.

(l) True. Studies show a positive correlation between higher levels of problem solving abilities and reading, as well as computational abilities. However, at least part of this influence may be the result of the correlation of these abilities with IQ. Studies do suggest that activities emphasizing reading skills such as drawing inferences, following directions, sequencing ideas, and identification of main ideas may improve problem solving abilities.

Factors that make problem solving more difficult are:

• *extraneous and irrelevant data.*	(m)	T	F
• *placing the question at the end rather than at the beginning of the problem.*	(n)	T	F
• *use of diagrams, graphs, or other visuals.*	(o)	T	F
• *adherence to a multistep problem solving model.*	(p)	T	F

(m) True. Problems with too much data and problems with too little data (for which the solver must supply additional information) are more difficult than problems limited to essential information.

(n) False. Placement of the question in a problem can affect the amount of time required for its solution, but it does not significantly change the achievement.

(o) False. The use of visual aids and manipulatives make a problem easier to solve.

(p) True. While structured problem solving techniques such as models offer some advantages, rigid insistence on their use does not appear to be effective. However, systematic instruction in procedures which may be helpful in problems, does aid children in developing problem solving techniques. Instruction which focuses upon the selection of the correct process to use and "why" it is appropriate is more effective than obtaining answers without such discussion.

* * *

Reasons frequently cited by researchers for children making mistakes in solving problems include: insufficient computation skills, lack of understanding of concepts, reasoning errors, weak vocabulary and failure to note details.

Some researchers have identified common traits that good problem solvers possess, hypothesizing that problem solving capabilities in others might be improved by attempting to improve those traits. Among the factors which characterize good problem solvers, those listed in the following table are the most frequently cited.

Traits Good Problem Solvers Usually Possess

- good estimation and analysis skills
- ability to perceive likenesses and differences
- reflective and creative thinking
- ability to visualize relationships
- strong understanding of concepts and terms
- ability to disregard irrelevant data
- capability to switch methods easily, but not impulsively
- ability to generalize on the basis of few examples
- ability to interpret quantitative data
- strong self-esteem
- low test anxiety

A common factor which many researchers cite as making an important contribution to problem solving development is the frequent opportunity to solve good quality problems which can be solved by children in a variety of ways.

Psychological Points and Teaching Implications

A number of eminent psychologists and expert problem solvers have studied and spoken on conditions affecting the development of problem solving ability. This section attempts to summarize some of the key points suggested by these authorities and to discuss implications for classroom instruction.

Jean Piaget and William Brownell

According to Piaget, problem solving growth is influenced by two sets of factors: (a) highly personal and narrowly individualistic perceptions associated with early childhood, and (b) social factors imposed on children which cause them to gradually substitute objective reality for subjective schemas, replacing illogical, personal mental processes with impersonal logical and rational thinking. Children are perceived as gradually passing from an egocentric to a socially aware thinking structure.

Prior to age seven or eight, reasoning is a mental experiment characterized by discontinuity, logical inconsistency, absolutism, juxtaposition, and syncretism. Until this age, Piaget's findings indicate that children find it difficult to solve problems wholly on a verbal basis and need to rely upon concrete materials which they can manipulate and associate with ideas in the problem. Not until the age of 11 or 12 does Piaget find children to be skillful in verbal problem solving and capable of adult kinds of reasoning.

Brownell, while supporting Piaget's description of how children develop power in problem solving, expands upon and modifies some of Piaget's findings. Brownell asserts that children at any age, not necessarily those older than seven or eight, are capable of solving problems suited to their experience, understanding, and behavior equipment. However, their processes may be largely nonverbal. In emphasizing the importance of problems being within the capabilities of children, Brownell cautions teachers against expecting too much of young children at verbal levels. "It is futile, perhaps dangerous, to ask children to solve problems which are beyond their powers or to solve problems by exclusively verbal processes before they can do so." [Brownell. 1942, p. 430]

Alluding to Piaget's levels of child development, Brownell emphasizes that these age levels are tentative and that children do not move from level to level in an all-or-none way. Children at any age may possess characteristics linked with several levels of thinking as they work with different kinds of problems. Moreover, Brownell questions the changes that occur in children's thinking structures as being the sole product of age differences. He suggests that increases in general experience and language facility may be just as important and that older children have merely had more opportunity for this development.

Turning to means to improve problem solving ability, Brownell criticizes a number of educational practices. He faults the step analysis techniques (1) "What is asked? What am I to find?" (2) "What is given? What do I know?" (3) "What process or processes should I use?" (4) "What is the probable answer?" He states that such an analysis does not necessarily characterize expert thinking by adults, and furthermore, that no research supports its being characteristic of good thinking by children. He further questions reliance on this technique to the possible exclusion of other essentials in problem solving such as facts, meanings, and understandings.

A second practice that Brownell questions is that of teaching 'cues,' the association of certain operations or processes with key words (e.g. "of" means multiply in percent problems). Not only is such advice fallible, but the more serious result of such teaching is that it may mask the failure of the teacher to develop a more fundamental and complete understanding of the operation, process, or problem situation.

A third practice with which Brownell advises caution concerns the teaching of skills more or less related to problem solving (e.g. strategies). He points out that such instruction may prove beneficial if the child lacks these specific capabilities, but that proficiency in these skills will be unproductive to a child lacking in basic understandings and rich concepts.

Commenting on the role of repetitive practice in improving problem solving, Brownell states that practice increases the facility to produce the practiced response for a given situation. However, he points out that practice on the same solution with the same problem does not increase the likelihood of that particular method being available for other problems, even those very similar. In fact, such practice might even limit the availability of the solution. But practice in using the method of solution and applying it to different problems results in increased probability that such a practiced method will be available in other problems.

Brownell offers his own recommendations in "Practical Suggestions for Developing Ability in Problem Solving" [Brownell, 1942, pp 433-40]. The following is a partial list of these suggestions.

1. For a learning situation to be a problem, the relationships required in its solution should be (a) well within the understanding of the child, and (b) identifiable with reasonable effort.
2. Skill in problem solving is a matter both of technique and meanings and understandings. Formal and abstract techniques should not be imposed upon children but should result in steady progress as end-products of development. Teaching should start with the techniques that children already use proficiently and progress through careful guidance toward more mature types of problem solving.
3. Meanings and understandings essential to problem solving cannot be given to children. They must be acquired through a wide variety of appropriate experiences. These meanings and understandings are most useful in problem solving when they have themselves been acquired through the solving of problems.
4. Experience which is most valuable for solving problems is *organized* experience. To the limits desirable and possible, solutions to problems should be summarized clearly, stated verbally, and generalized.
5. Practice in solving problems should not consist in repeated experiences in solving the same problems with the same techniques, but rather in the solution of different problems by the same techniques and in the application of different techniques to the same problem.
6. A problem is not necessarily 'solved' because the correct response has been made. A problemt is not truly solved unless the learner understands what he has done and knows why his actions were appropriate.
7. Mistakes which children make in trying to solve problems are not 'corrected' by providing them with the 'right' solution. They are corrected only when the weakness in technique has been exposed and supplanted by a more appropriate method or when the required meaning or understanding has been developed.
8. One aspect of problem solving is the ability to differentiate between reasonable and absurd, logical and illogical results. Instead of being 'protected' from error, the child should at times be exposed to error and encouraged to detect and to demonstrate what is wrong, and why.
9. A problem-solving attitude, an inquiring and questioning mind, is possible of development. It does not occur by accepting from others truths and conclusions which ought to be established by the learner himself. The attitude *is* produced by continued experience in solving real problems, one consequence of which is that the learner comes to *expect* new problems and to look for them.
10. Suggested activities for guiding learning in a particular problem task:
    a. Help children to formulate the problem clearly.
    b. See that they keep the problem continuously in mind.
    c. Encourage them to make many suggestions by having them analyze the situation, recall similar cases and the rules or principles of solution which are applied, and guess courageously.
    d. Get them to evaluate each suggestion. This involves maintaining a state of suspended judgment, criticizing the suggestion by anticipating objections and consequences, and verifying the conclusions by appeal to known facts, miniature experiments, and authorities.
    e. Have them organize their process of solution by building outlines, using diagrams and graphs, taking stock from time to time, and formulating concise statements of the net outcomes of their activity.

Norman R. F. Maier

Maier spent many years studying general problem solving abilities for individuals and groups. Mathematical situations frequently were the vehicles for his investigations. One of the many areas that he explored was what makes a problem difficult. Ruling out lack of knowledge and limited mental powers, he has identified several areas of difficulty in problem solving [Maier, 1970, pp, 179-188].

1. Misleading incorrect solutions.

Some problem solvers fail to realize when they have an incorrect solution. Therefore, they stop further effort on the problem. Problems vary in their capacity for yielding incorrect plausible alternatives to the correct solution and in the ease with which incorrect solutions can be identified.

R_x this difficulty (suggestions are not Maier's): Looking back problem solving strategies stress that a problem is not necessarily solved because an answer has been achieved. The problem solver should consider whether his answer is reasonable, what implications it has to other situations, how it varies if conditions are changed. Such actions may reveal faulty logic if an incorrect result has been chosen or extend understanding about the problem if the correct answer has been determined.*

2. Difficulty in selecting from given alternatives.

In some problem situations, the solver is expected to select the correct alternative from a group of possible solutions. The difficulty rests in the correct choice to be made. A multiple-choice test is an example of this kind of situation. Choice of the correct response can be made more or less difficult by making the incorrect responses more or less plausible or by decreasing the differences between the alternatives.

R_x: The ability to discriminate among possible alternatives is a valuable life skill. However, its use in developing problem solving should be handled very carefully. In a multiple-choice setting, the problem solver may elect either of two courses of action. He may employ individually-devised selection criteria to identify incorrect alternatives until only one answer candidate is left. Or he may temporarily ignore the available solution candidates while he pursues the correct answer through independent actions.

Skill in the first problem solving technique might be expedient when the correct answer is known to be among the alternatives. For example, use of this skill on multiple-choice instruments might improve test scores. However, the correct solution may not be listed among the alternatives. Elimination of incorrect choices indicates to the problem solver that the correct answer is not present, but this technique will not necessarily be of help in finding that answer. Therefore, a wide variety of methods and procedures are needed for overall problem solving development and evaluation.

3. Unusual response required and low priority in the behavior repertoire.
   (Maier lists these difficulties separately)

Success in generating the correct response depends upon how long it takes for that alternative to be considered and tried. When the correct response lies outside the common experience or thinking patterns of the solver, the problem becomes more difficult and lengthy in solution for that individual. Assignment of low priority to a response can be the result of differences in experience levels, biological makeup, or subjective judgments by the problem solver as to what is permissible or desired. *Trick* questions and *insight* problems frequently involve this kind of difficulty.

R_x: Experience levels can be broadened and individual differences due to backgrounds ameliorated by providing a rich variety of problem solving activities. Coupling these activities with a classroom environment which values unusual or innovative approaches, alternative solutions for the same problem, and a strategies approach to problem solving can expand individual behavior repertoires.

4. "Idea-getting" versus "idea-evaluation."

Reaching a final solution depends upon both idea-getting (generating alternatives) and ideal-evaluation (choosing the best alternatives). Sometimes we do not achieve a satisfactory solution because we put too little effort into considering alternatives or we make a poor selection from those evaluated. Frequently, the obstacle to successful problem solving is the tendency to evaluate and select an alternative before better ideas have been generated. Unless idea-getting is stressed and idea-evaluation temporarily suppressed, the presence of available alternatives can impede the possible consideration of other, more viable alternatives.

*Looking back strategies are discussed in the next section.

R_x: Students need to develop the ability to suspend judgment in the early stages of solving a problem, setting aside the known alternatives and exploring the possibility that less familiar alternatives may exist. Solving the same problems by different methods can help them appreciate the fact that solutions may be achieved by less-easily identified strategies or approaches.

5. Locating surmountable obstacles.

It is a common failing for people to concentrate upon the first obstacle identified. Regardless of whether this obstacle is easily or not easily surmountable, effort continues to be expended in attacking the problem from this perspective. In many instances, it is far more productive to identify alternative obstacles to problem solution which may be more easily overcome.

R_x: Problem solving expertise consists of skill in identifying obstacles that can be easily circumvented and of ingenuity in dealing with particular obstacles. The identification of problem obstacles is generally given too little priority because we are "solution-oriented." Too little time is spent exploring the problem situation. To help students draw distinctions between goals and obstacles, each should be clearly differentiated and carefully considered before any additional problem solving activity is undertaken.

6. Motivation.

Motivation is frequently an intrinsic part of the challenge of a particular problem situation. However, individuals vary in their response to specific challenges. Fear of failure, need for achievement, and the stage at which an individual gives up a solution search are personality factors that influence the motivation that a particular problem situation presents.

Variables associated with the degree of motivation experienced by individuals are highly subjective, very complex, and difficult to measure. Strong motivation can be a facilitating anxiety but it can also act as a deterrent to problem solving. "Strong motivation may even serve as a handicap in that it may generate 'solution mindedness' at the expense of 'problem mindedness.'" [Maier, 1970, p. 187].

R_x: The perceptive teacher will recognize the contribution of motivation to problem solving success as well as its subjective nature. Variety in problem situations, formats, and degree of difficulty are ways to accommodate individual differences. Alert for variation of student responses to particular problems, the observant teacher can provide appropriate individual direction.

7. Degree of stress.

Pressure to produce a solution to a problem generates stress that may have either positive or negative influences on problem solving. Whether such stress results in improving effort or increasing anxiety depends on whether the frustration threshold is exceeded. Some individuals appear to perform better under stress while others go to pieces. The matter can be further complicated by the emotional involvement of the problem solver (for example, one person can frequently solve another's problem while not able to solve his own).

R_x: Recognizing that individuals perform differently under stressful limitations such as time pressures or reward systems alerts the resourceful teacher to using varying degrees and kinds of stress to suit individual needs. For example, a teacher might alleviate some of the stress for a strongly motivated but temporarily frustrated student by guiding him temporarily away from the source of his frustration. Under less self-imposed pressures, the student may favorably respond to the same challenge. For another student who feels keenly the need to be successful for some extrinsic motivation such as grades, a teacher may reduce this anxiety by assuring the student that he need have little concern for such matters regarding his present activity.

Maier summarizes his findings on problem difficulties by concluding that high ability in solving some problems is not necessarily a good predictor of success with different kinds of problems. The obstacles and difficulties posed by problems are highly subjective in terms of how individual solvers interact with the problems. He associates expert problem solving with the traits of proficiency with a variety of techniques, a knowledge of types of common errors, a high frustration threshold, constructive motivation, and flexibility in varying problem approaches.

Herbert A. Simon and James Greeno

Simon and Greeno represent a group of modern psychologists who are studying thinking and learning in terms of information processing approaches. The brain is modeled as a computer with processor, short and long term memory, coding systems, search mechanisms, and the like.

Simon points out that students can be induced to learn quite different things about the same task. With respect to problem solutions, he quotes a Katona experiment [Simon, 1976, p.8] which shows that solutions are learned best in the order: (1) intuition, (2) logic, (3) rote. "Best" in this context refers to more quickly, better retention, and producing greater amounts of transfer.

Two weaknesses that Simon identifies in problem solving are recognition of salient problem features and the search for good problem representations. Good problem solvers tend to emphasize certain elements of a problem situation while deemphasizing others until a later stage in the problem. Simon notes that good problem solvers [illegible] notice relevant problem data and associate them with promising solution steps. In doing so, the solvers often unconsciously employ notions of promising search paths.

"The deliberate search for a good problem representation, or even a capability for generating and considering alternative representations does not seem to be a common part of the human problem solving repertory" [Simon, 1976, p. 18]. Simon concludes that this is a skill in which most students do not get explicit instruction.

Greeno discusses attempts to increase children's problem solving skills and offers the following advice.

1. One should not think of problem solving skill as a single, uniform capability. Problems of different kinds may require substantially different problem solving skills.

2. For problem solving instruction to be successful, training has to have specific objectives, something which students should be able to do at the end of the training that can be identified by them as well as by objective assessment.

3. Problems solving does not occur in a vacuum. Students accomplish solutions to problems by using knowledge that they have about the problem situation or about a similar situation from which they can transfer their knowledge. Even when problem solving uses very general skills, it is dependent upon this use of specific information. "Thus, it is useful to try to identify information that would be useful to students across a variety of problem solving situations, and teach them that information, as well as give them experience in using the information in solving problems" [Greeno, 1977, p. 18].

Greeno goes on to identify four general aims for an instructional program of problem solving: (a) strengthen students' skills regarding the kinds of problems they are expected to solve in the ordinary curriculum; (b) enlarge the domain of problems that students are able to work on by providing them with additional knowledge that they have available in problem solving; (c) expand the range of skills that students have by giving them training that involves different kinds of problems from those they ordinarily encounter in school, though still related closely to the school environment; and (d) identify and analyze problems that are clearly nonacademic, such as personal and social problems, and teach approaches to these problems.

Greeno is critical of the imbalance in school activities in favor of problems for which there is a correct answer. He suggests that teachers supplement the curriculum with problem situations in which no single correct answer is available, and encourage students to generate a variety of fruitful responses. Toward this end, he argues that a major effort in problem solving instruction is to encourage positive attitudes toward having ideas, particularly those which seem unusual or fanciful. Encouraging imaginative thought, he stresses the importance of getting students to postpone critical judgement of their ideas.

Grayson H. Wheatley

Wheatley is one of several researchers specializing in how the brain functions with respect to thinking and reasoning capabilities. Current research has concluded that the two hemispheres of the brain have specialized functions. For most persons, the left hemisphere is used in performing routine sequential tasks, logical reasoning, and analysis of stimulus components. It also is the brain's language center. Stimuli are processed in a way that such information can be expressed in language. The right hemisphere processes stimuli holistically rather than sequentially. It operates in images rather than in words. Tasks such as spatialization are carried out in the right hemisphere.

The implications of hemispheric specialization to problem solving have made us aware of the separate contributions made by each part of the brain to the solution process. The left hemisphere's language capability processes the problem information (particularly when in story form) and breaks the problem down into components. This hemisphere performs the analytic functions on the problem. If rule application or algorithmic action are called for, the left hemisphere is used.

The right hemisphere carries out synthetic problem solving activities. Problem information is considered and restructured holistically. Original or creative problem thinking originate in the right hemisphere as well as reflection about a problem once an answer has been reached.

Wheatley [1977, p. 37] cites studies by Bob Samples of Evergreen State College that conclude that children pass through three stages in solving a problem: (1) They appear to consider the problem in nondirective ways, mulling it over, thinking by analogy. They may give the appearance of disinterest or not trying but are in reality formulating a plan for solution. (2) Then, their actions change very markedly. Movements quickening, they proceed rapidly to the application of the selected method, frequently use of a previously learned rule or algorithm. (3) They reflect on the problem.

In the first phase of this thinking sequence, the right hemisphere leads the left. At the second stage, the left hemisphere is used predominantly, and at the third level, the right hemisphere may again be brought into action. The consequence of these stages and their interactions with the two specialized hemispheres of the brain suggest the need for realistically directing children's expectations at each stage to appropriate activities. For example, a child may become very frustrated at level one if he perceives the first stage to be primarily a rule or algorithmic search. Without the required analysis of the problem elements and restructuring into promising avenues, a satisfactory plan is not likely to emerge.

Wheatley concludes that present mathematics curricula are unbalanced in favor of left hemisphere activities and offers several suggestions for encouragement of more right hemisphere use. He suggests use of puzzles (particularly of a spatial nature), tasks that are open-ended and which require investigation, laboratory and discovery methods, increased use of problems with multiple solutions or requiring alternate methods of solution, and activities that entail problem formulation.

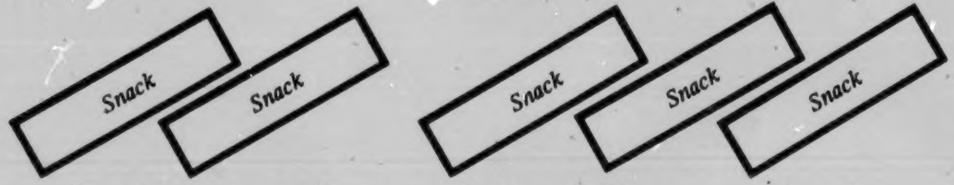
Wayne Wickelgren

Wickelgren has focused a portion of his professional interests on the study and teaching of problem solving approaches. He describes a variety of general actions which can be utilized in the solution of problems (e.g. hill-climbing, getting out of loops, defining subgoals). An intriguing characteristic which he identifies as distinguishing experienced from less-experienced problem solvers is that of recognizing the equivalence of alternate solution paths.

Wickelgren defines a *microaction* sequence to be a sequence of actions that starts with a given state and achieves some desired state. There may be more than one such sequence that will produce the desired state from the given state. The collection of all such action sequences is an equivalence class which Wickelgren refers to as a *macroaction*.

To illustrate this difference, consider the following problem.

Grades 3–4 *Candy Bars*



2 for 25¢ *Which is the better buy?* *3 for 40¢*

The average third or fourth grader is likely to approach this problem by transforming the price for multiple bars to an equivalent cost per single bar. Then a cost comparison can be made for single bars. An older or more experienced child (or adult) might perceive the problem from a broader perspective. That person would recognize that such a cost comparison depends upon a ratio of cost per bars or bars per cost and that essentially a comparison of ratios is involved. Prior experience with ratios would indicate that (1) ratios may be transformed into equivalent but different forms, and (2) that comparison of two ratios depends upon equating one pair of corresponding parts while comparing the remaining parts of the ratios.

In the given problem, we could proceed in two different ways with variations of each.

Reduction to unit quantities:

($\equiv$ means "is equivalent to")

(a) $\frac{25¢}{2\text{ bars}} \equiv \frac{12\ 1/2¢}{\text{bar}}$ and $\frac{40¢}{3\text{ bars}} \equiv \frac{13\ 1/3¢}{\text{bar}}$

Conclusion: The cost **per bar is cheaper** at 2 bars for 25¢ so that is the better buy.

(b) $\frac{2\text{ bars}}{25¢} \equiv \frac{0.08\text{ bar}}{1¢}$ and $\frac{3\text{ bars}}{40¢} \equiv \frac{0.075\text{ bar}}{1¢}$

Conclusion: The **amount** of candy bar **per cent is more** at 2 bars for 25¢, so that is the better buy.

Expansion to identical quantities:

(c) $\frac{2\text{ bars}}{25¢} \equiv \frac{6\text{ bars}}{75¢}$ and $\frac{3\text{ bars}}{40¢} \equiv \frac{6\text{ bars}}{80¢}$

Conclusion: The **same number** of candy bars can be **purchased cheaper** at 2 for 25¢ so that is the better buy.

(d) $\frac{2 \text{ bars}}{25¢} = \frac{16 \text{ bars}}{\$2.00}$ and $\frac{3 \text{ bars}}{40¢} = \frac{15 \text{ bars}}{\$2.00}$

Conclusion: **More bars** can be purchased **for the same money** at 2 bars for 25¢ so that is the better buy.

For this problem, a microaction entails changing a given ratio into a form suitable for comparison. An equivalence class of microactions consists of the four actions resulting in changing the given ratios into another form for a particular kind of comparison. Each method achieves the goal condition. Therefore, they represent equivalent actions, a macroaction, with respect to solving this problem.

The *macroscopic* thinker perceives equivalent actions (not necessarily all of them) and is able to select one by some additional criteria such as "most efficient" or "easiest to accomplish."

Consider the task of solving the equation $2x - 7 = 5(x + 4)$. A number of differently sequenced steps are possible to produce the sought-after goal state of $x = k$. The macroscopic thinker realizes that each sequence of actions is equivalent to every other in producing this end. He chooses one and carries through the sequence of microactions in a very routinized way. An eighth or ninth grade student is not likely to possess such insight. Microscopically, he must think through each step of the solution.

Another way that a problem solver may use macroscopic thinking is in the size of the information units that he thinks. Consider, for example, a difficult geometric construction whose solution may utilize several more elementary constructions. Macroscopic thinking about the problem perceives single actions as discrete simpler constructions (a perpendicular to a line through a point, a line parallel to a given line through a given point) rather than the microscopic swinging of arcs and drawing of lines.

Everyone uses macroscopic and microscopic thinking to varying degrees in different situations. Macroscopic thinking depends upon content familiarity as well as problem solving skills. Exposure of students to certain well-worn paths that often prove to be productive is a major objective of problem solving instruction. Through such frequent and diverse experiences, students are likely to broaden both their content and problem solving expertise while improving their capability to utilize macroscopic thinking.

As teachers, it is necessary to realize that no one utilizes macroscopic or microscopic thinking all of the time. Even moderately experienced problem solvers may perceive a problem situation at microscopic levels for a particular discussion. Conversely, learners exposed to new problem situations may proceed very quickly to insightful understanding of underlying concepts and principles. The perceptive teacher will (a) ascertain which students are viewing a given problem on a particular level, and (b) direct his comments and teaching to that need. Using a macroscopic explanation with students who are on a microscopic level is self-defeating and frustrating. Directing a microscopic explanation to macroscopic thinking individuals is pedantic and dull.

Research on problem solving as well as psychological descriptions of how it occurs represents an incomplete science at best. Though a great deal of information has been gleaned about factors which affect and influence problem solving, a clear and comprehensive structure which accounts for all of the known data, as well as individual differences, has not yet emerged. However, a great deal is known that we can make use of in creating conducive environments and learning situations in which problem solving development can be nurtured.

LOOKING BACK STRATEGIES

"Nothing is more important than to see the sources of invention, which are, in my opinion, more interesting than the inventions themselves.

–Gottfried Wilhelm Leibniz

The following strategies are called "looking back strategies" since they are applied after the answer to a problem has been achieved. We have chosen to discuss them in booklet two because they are more difficult for students to learn than those strategies presented earlier. Introduction of these strategies at the elementary level is enhanced by emphasizing their use in group rather than individual work. Through class and small group discussions, students may witness the advantages of *generalizing, checking the solution, finding another way to solve it, finding another solution, and studying the solution process.* With appropriate teacher encouragement, they will grow in their abilities to apply these "looking back strategies" to their own work.

Strategy: Generalize

Generalizing can enter the problem solving process in two distinct ways. We may first use generalizing to arrive at a specific solution to a problem. In understanding and beginning to work on the problem, we may glean from it certain elements which we recognize as fitting a more general class of properties or relationships, such as "distance and time can be used to compute a rate." We may deliberately choose to solve a more general problem than the one posed. Or we may temporarily suppress specific problem data to better understand and use relationships. For example, "If I wanted to go anywhere in a certain amount of time, I would have to determine a rate that is sufficient to cover the distance in the given time; that means . . ."

After a solution to a particular problem has been achieved, we may use generalizing in a second way – to extend our conclusions to more general and far-reaching situations. It is this second application of generalizing on which we wish to focus as a looking back strategy – making children aware that obtaining an answer is not necessarily the end of a problem. Reexamination of the problem, the result, and the way it was obtained will frequently generate insights far more significant than the answer to the specific situation. Experiencing these insights, the learner can make observations that will enable him to solve whole classes of similar and perhaps more difficult, problems.

> "One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else."
>
> – George Polya

Consider the following problem.

Grades 5 & 6 *Only Three Weights*

> *A boy selling fruit has only three weights and a double pan balance. But with them he can weigh any whole number of pounds from one pound to 13 pounds. What weights does he have?*

Suppose that a class has individually or collectively solved this problem and arrived at the solution of 1-, 3-, and 9-pound weights. As teacher, you realize the potential this problem has for being extended. So you ask, "What would be a wise choice if you could buy a fourth weight? How many additional weighings could you make with four weights?"

If you would like to experience the additional insight that comes with generalizing this problem to other cases, stop reading and try to answer the questions yourself.

Perhaps a fruitful way to begin is to back up and examine how we get from one weight to two and then to three. If we had a weight of 1 and could add another, why did we choose 3 instead of 2? The answer is that we can weigh amounts of 2, 3 and 4 by picking 3.

Selecting a third weight in order to weigh objects up to 13, we are restricted by the knowledge that 4 is the maximum amount that can be weighed with two weights. Therefore, 1 + 3 + W = 13, implies W = 9.

The following table shows how we may obtain all weights 1 to 13.

Wt	Combination	Wt	Combination	Wt	Combination
1	1	5	9 - 3 - 1	10	9 + 1
2	3 - 1	6	9 - 3	11	9 + 3 - 1
3	3	7	9 + 1 - 3	12	9 + 3
4	1 + 4	8	9 - 1	13	9 + 3 + 1
		9	9		

(weights subtracted are placed on the pan with the object).

As we study the procedures used in the above table, we notice that: (1) the maximum amount that can be weighed by any combination of weights is achieved by placing all weights on the same pan (adding weights); (2) the next higher weight above that maximum is obtained by using a new, heavier weight on one pan and placing all smaller weights on the opposite pan (subtracting all smaller weights, in effect). For example, using 1 and 3, the maximum that can be weighed is 4 by placing 1 and 3 on the same pan. The next weight 5 is obtained by introducing a new weight 9 and subtracting all smaller weights, 9 - 1 - 3 = 5.

Suppose we try the same technique with a fourth weight Z to achieve the next possible weight beyond 13 (our present maximum). Then,

$$Z - 9 - 3 - 1 = 14$$

Therefore, Z = 27

After constructing an expanded weight table, we can convince ourselves that we can indeed weigh any object from 1 to 40 (27 + 9 + 3 + 1) with the four weights, 1, 3, 9, 27.

Now we might make a table contrasting the number of weights, the largest weight, and the maximum weight possible to be able to generalize the solution even further.

No. of weights	Largest Single Weight	Maximum Possible Weight Combinations
1	1	1
2	3	4
3	9	13
4	27	40

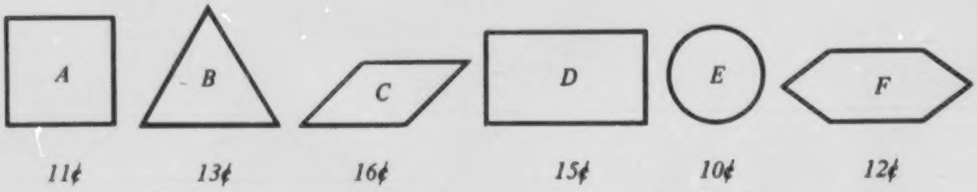
With a short inspection, we can successfully predict the sizes of additional "largest single weights" as successively larger powers of 3 (or 3-multiples of the previous largest single weight): 81, 243, . . . "Maximum possible weight combinations" are the sum of all weights used or simply the new, largest weight added to the previous maximum weight combination. Therefore, maximum possible weight combinations with five and six weights are respectively 121, 364 . . .

The possibilities for continuing to generalize are far from exhausted. We could try to develop an expression for the largest weight in terms of the number of weights used, for example. Moreover, we have not discussed some of the generalizations discovered in manipulating weights to produce combinations between 13 and 40. Clearly, we could continue to extend this problem beyond the capabilities of the original 5th-6th grade students for which it is recommended. However, we feel that the point has been made that examining the solution to the original problem and attempting to generalize it, to extend it slightly, creates a great deal of further insight. The amount of extension and the way that it is encouraged, whether through group work or individual challenge, as two examples, is a teaching decision.

Using the strategy, *generalize*, we can take a somewhat easier problem and expand our understanding of it more quickly.

Grades 3-4 *Money to Spend*

Morris has exactly 26¢ to spend. He wants to spend all of his money, and he has the following toys to choose from. What can he buy?



Children will make and use many generalizations in arriving at their answers to this problem without consciously being aware of these considerations. As the teacher, you can bring out these generalizations with questions such as:

"How do you go about determining the toys that Morris could buy? Did you have to check all combinations?"

"Suppose you had to answer another question similar to this one but with different numbers. How would you use what you learned about this problem to help you with the new problem?"

Or the questions can be even more specific and directed:

"Suppose the prices are such that you cannot buy more than two toys. If the amount of money that you have is even, how does this affect the toy combinations that you look at?"

"If one toy costs less than half of the money you have, what must the price be of the second toy you buy?"

Getting children to focus on the relationships involved in a problem, and consequently, generalizations of the problem, can sometimes be practiced by giving problems without numbers.

Grades 4-5 *No Numbers Tennis Store*

NoFault Tennis Store has tennis balls for sale by the box. For the amount of money that Suellen has, she can buy a certain number of boxes. What price does she pay per ball?

Having students describe how they would solve such a problem, listing the information that they need, will draw more specific attention to relationships in the problem.

Strategy: Check the Solution

It may seem strange to think of checking a solution as a strategy. After all, a problem is really not solved until you find an answer that satisfies the requirements of the problem. Yet, the very length of a problem or the fact that the original problem becomes symbolic in the process of solution may make us lose sight of the original problem that we were attempting to solve. Since it is a very human frailty, particularly with textbook problems, to stop at the numerical answer without referring back to the original problem, we feel that it is properly a looking back strategy.

Grades 7-8 *Five Numbers*

When five consecutive numbers are added together, their sum is 155. Find the numbers.

A good problem solver might say to himself that the average, 31, is likely to be the middle number with the other numbers paired, one higher and one lower than 31, to produce 29, 30, 31, 32, and 33 as the answers.

But let us suppose that such a problem has been given shortly after work involving the use of variables to represent unknown quantities. Then a student might let N equal one of the numbers, N + 1 the next and so on. His solution might then look like,

$$\begin{aligned} N + (N + 1) + (N + 2) + (N + 3) + (N + 4) &= 155 \\ 5N + 10 &= 155 \\ 5N &= 145 \\ N &= 29 \end{aligned}$$

The original problem becomes a symbolic problem and requires concentration to manipulate the symbols correctly. Attention is focused away from the actual problem and the quantities which the symbols represent. Unless the student checks this "answer" against the requirements of the problem, he may very well write N = 29 as the answer to the problem.

A more direct advantage to checking the solution is in finding errors in computation or reasoning that have produced a result that does not satisfy all of the problem requirements.

Grades 3-4 *Donuts and the Party*

Mrs. T. Leaves purchased six dozen donuts for the 16 ladies of her sewing group. All the donuts were eaten. How many donuts did each lady eat if each ate the same number?

One youngster might obtain an answer of 1152 (that is 6 x 12 x 16). Without considering the implications of this result (even on a human scale of consumption), he might blithely go on to the next problem, confident that he was correct.

This introduces a very important aspect of checking the solution, that of **determining the reasonability of the result**. Checking the solution does not mean that one must (or even should) go through all of the calculations again. Nor does it necessarily imply that one must determine whether his answer **exactly** fits the given conditions of the problem (although that is the only fail-safe thing to do). Sometimes, it is sufficient to determine whether the answer is "in the right ballpark" in answering the question posed.

Grades 6-7 *Scouts and Cookies*

The local scout troop sold 4000 boxes of cookies last year. This year, they need to earn $1600. If they sell the same amount of cookies and cookies cost $1.25 per box, how much should they charge per box?

A series of calculations involving 1600 x 1.25 = 2000 and 4000 ÷ 2000 = 2.00 might suggest to the unwary student that the price per box should be $2.00. A simple check of the calculations will reveal that all calculations are correct!

But checking the solution also means determining whether the answer makes sense. The careful problem solver will determine the reasonability of the answer by a different means than that used to generate the answer. This decreases the likelihood of making the same mistake. For example, if $2.00 is charged per box, the profit will be $0.75 per box or 4000 x $0.75 = $3000 total. This amount is far in excess of what they need. Obviously, a reexamination of the solution is in order.

Strategy: Find Another Way To Solve It

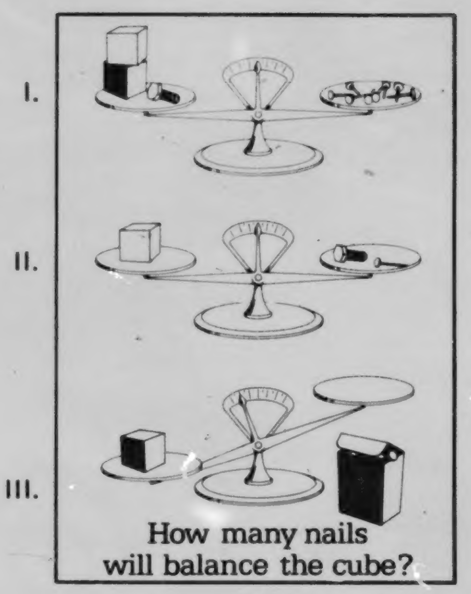
Nearly everyone is faced with coping with a real-life task that he must handle periodically. Seldom does one continue to resolve this task in the same way that he first accomplished it. In most instances, he finds different – often substantially different – and almost always better ways to confront and deal with the situation.

The same potential exists in problem solving when facing a **good problem**. A good problem can be approached and solved in many different ways, each contributing new insight and additional understanding of the problem situations. The goal of problem solving is to study and develop processes that lead to the solution of problems. Discarding a good problem just because an answer has been found can be a foolish and wasteful action. Good problems can be compared to works of art. They contain many facets that are not yielded to cursory inspections. They warrant careful study and reexamination. Such effort is rewarded with new discoveries and insights about that particular problem as well as whole clusters of unsuspected ideas and relationships.

But first one has to look.

Grades 3-10 A Weighty Dilemma

Solve this problem in at least two different ways before reading on.



How many nails will balance the cube?

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The following solutions were contributed by three different age children, but could easily have come from a single individual if he had the challenge and opportunity to look for more than one solution.

9 year-old solution. (replacement): A cube equals a bolt plus nail (II). So replace a cube in (I) with a bolt and nail. Remove a nail from each side. There are now 1 cube and 2 bolts opposite 7 nails. Replace the second cube in (I) with a bolt and nail. Remove a nail from each side. Now 3 bolts are opposite 6 nails. So a bolt equals 2 nails. Now look at (II). If a bolt is equal to 2 nails, the cube weighs 3 nails.

10 year-old solution. (guess and check): There are three objects in (I) balanced by 8 nails. Too bad, there aren't 9 nails. Then I would guess 3 nails as the weight of each object. Let's see, cubes are larger than bolts so they probably weigh more. Suppose I do guess 3 nails for a cube. What does that mean in (II)? Then 3 nails balance 1 bolt and 1 nail. That means that the bolt weighs 2 nails. Let's see if that checks. Two cubes is 2 x 3 = 6 nails plus 2 nails for the bolt; 8 nails equals 8 nails. It checks!

13 year-old solution. (elimination): To answer the question, I need just cubes on one pan and only nails on the other. So in (I), I need to get rid of the bolt. Looking at (II), I can replace a bolt and nail with a cube. So in (I) if I add a nail to each pan, the balance will be kept. Then, I can replace the bolt and nail on the left pan with a cube. 3 cubes will be balanced by 9 nails, so 1 cube equals 3 nails.

These three solutions of this problem are delightfully different and distinctive in the approach that each solver took to the problem. Three very worthwhile and important mathematical ideas have been unearthed and put to good use in resolving this dilemma. A much clearer and deeper understanding has to result for the problem solver who experiences (or discovers as part of a group) these different approaches. He will perceive more about balances, problem solving, intuitive equation solving, and other difficult to pinpoint insights that may be unique to his previous background and experiences.

The possible ways of approaching this balance problem are also far from exhausted. Inspecting the three given solutions, they all involve one common property – starting with condition (I) and trying to use condition (II) in condition (I) to simplify the situation and answer the question. *We generalized.* Suppose we reverse this order. Looking at (II), we could try to eliminate one type of object from this balance by using the relationships expressed in (I).

Solution 4. (We will use equations to represent the actions: C = cube, N = nail, B = bolt).

From (I) $8N = 2C + 8$. Increase the amounts on each pan of (II) eightfold.

Then

$$(II^{*}) \quad 8C = 8B + 8N$$

Replace 8N with 2C + B so that

$$(II^{**}) \quad 8C = 8B + 2C + B$$

27

Remove 2C from each pan,

$$\begin{aligned} (\text{II}^{***})\quad 6C &= 9B \\ \text{or}\quad 2C &= 3B \end{aligned}$$

Replace 2C in (I) with 3B so that

$$\begin{aligned} 3B + B &= 8N \\ 4B &= 8N \\ B &= 2N \end{aligned}$$

Then in (II) replace the bolt with 2N,

$$\begin{aligned} C &= 2N + N \\ C &= 3N \end{aligned}$$

Solution 5.

In (II) add 2 cubes to both pans, producing

$$(\text{II}^{*})\quad 3C = 2C + B + N$$

From (I) $2C + B = 8N$; replace $2C + B$ in (II*) with 8N.

$$\begin{aligned} (\text{II}^{**})\quad 3C &= 8N + N \\ \text{so}\quad C &= 3N \end{aligned}$$

Solution 6.

Add the pan contents from (I) to the **corresponding** pans of (II).

$$(\text{II}^{*})\ C + 2C + B = B + N + 8N$$

Remove one bolt from each pan.

$$\begin{aligned} (\text{II}^{**})\quad 3C &= 9N \\ C &= 3N \end{aligned}$$

Solution 7.

Double the amounts on each pan of (II)

$$(\text{II}^{*})\quad 2C = 2B + 2N$$

Add the pan contents from (I) to the **opposite** pans of (II)*.

$$(\text{II}^{**})\ 2C + 8N = 2B + 2N + 2C + B$$

Remove 2 cubes from each pan.

$$(\text{II}^{***})\quad 8N = 3B + 2N$$

Remove 2 nails from each pan.

$$\begin{aligned} (\text{II}^{***})\quad 6N &= 3B \\ 2N &= B \end{aligned}$$

Replace the bolt in (II) with 2N.

$$\begin{aligned} C &= 2N + N \\ C &= 3N \end{aligned}$$

In the course of seven alternate solutions of this problem (and we are confident that there are more), we have encountered the following notions:

a. substitution
b. estimation
c. equals subtracted from equals give equals
d. partitioning of sets
e. equals added to equals give equals
f. equals multiplied by the same number are equal

How rich an experience is missed by being answer-oriented and stopping with the first solution.

Strategy: Find Another Solution

Largely due to the kind of textbook problems they have experienced, students tend to approach many problem situations expecting one and only one correct solution. In many practical situations, there may be many answers that are correct or acceptable. At other times, the correctness of a solution may depend upon the operating assumptions or data that one accepts at the beginning of a problem. Consequently, students need experience with problems that have more than one solution (even when there is no indication that more than one exists) and with problems for which their answer is not likely to agree with their neighbor's because of different assumptions made.

Grades 4 - 5 *Pocket Change*

You have $1.00 in your pocket made up of seven coins. What are the coins?

Nothing in the problem statement suggests that more than one solution exists. Yet students should not complacently assume that their first solution is the only one, or that in finding a solution that they have satisfied the problem.

50¢	25¢	10¢	5¢	Total
1	1		5	100¢
1		4	2	100¢
	3	1	3	100¢
	2	5		100¢

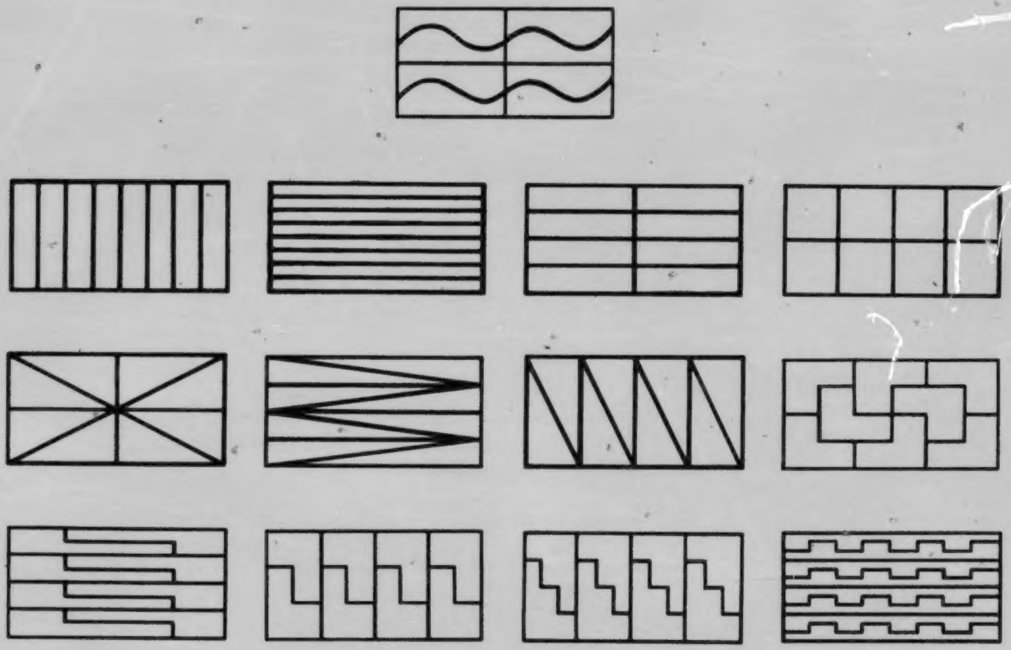
The process of convincing oneself that these are the **only** solutions in this problem requires a great deal more insight and problem solving understanding than stopping after any single answer above.

Young children in particular can profit from problem activities that demonstrate that more than one answer is possible. An open-ended problem like that below can become an exercise in creativity as well as problem solving.

Grades 2-3 *Cutting a Candy Bar*

In how many ways can you cut a candy bar into 8 pieces? All pieces have to be the same size and shape.

Some answers to this question are (as viewed from the top):



A problem which must be tailored to the individual student will make him aware that answers to the same question may vary according to initial conditions or assumptions.

Grades 4-5 *How Old Are You?*

Find out how many days old you are to this very day!

Students will be anxious to determine whether they are correct. Realizing that their answer will differ from their classmates, students will seek some verification by checking "how close" their answer is to their neighbors. A very good class discussion can evolve from the question: "How could we determine whether all your answers are correct?" Steering the discussion toward finding a range of possibilities by verifying the answers from the youngest and oldest class members is just one of many excellent ways that this problem solving activity could be extended and heightened.

Strategy: Study the Solution Process

The action of solving a problem, particularly a long or difficult one, may consist of several stages. At the very least, a number of steps, some of which may even lead away from the eventual solution, might be involved. In passing through these stages or steps, the learner will probably be aware of the route he is taking toward solution only in a secondary way at best. Upon determining the answer, he has reached one problem solving plateau; he has achieved the answer goal. But generally he will not have an overall perspective of the problem and the thinking and actions required for solution.

To help the learner reach this higher plateau of problem solving learning for given problems, you can request that he retrace and study the solution process in some manner. This will permit the student (and you) to identify the thinking used at each stage, the additional facets of the problem that were unveiled, strategies that were formally employed, and actions that were nonproductive.

Studying the process of solution puts a problem into perspective. It makes the activity a more complete learning experience than answer-getting. Concepts and relationships involved in the problem emerge and fit more logically into a student's personal mental structure. Coupled with questions such as, "How would the solution be changed by . . .?", *studying the solution process* can expand an individual problem into a rich and meaningful total view of a family of related problems.

Inducing students to apply this strategy can be achieved in several different ways. The followlist contains some valuable review activities that require a reconsideration of the total problem.

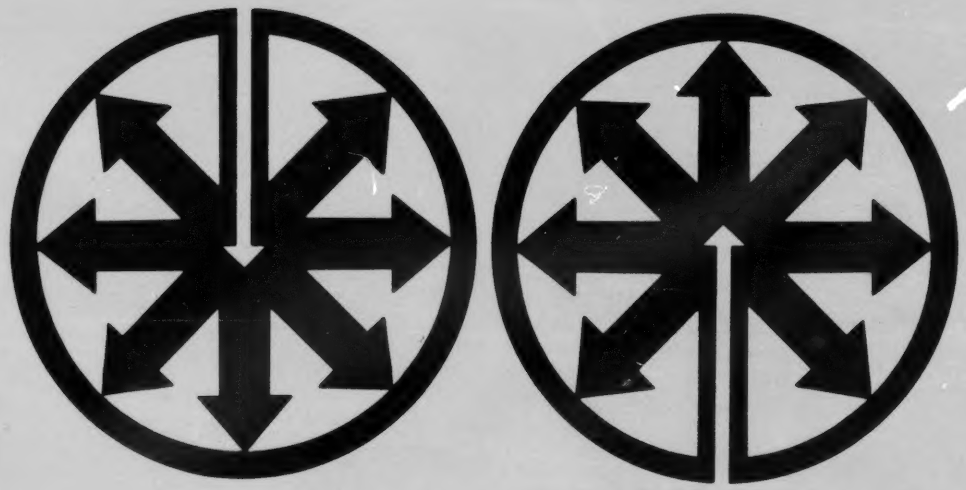
Activities Requiring Study of the Solution Process

1. Ask a student to verbally describe to the class how he solved the problem (including the why steps).
2. Have a student teach the solution of this problem to another student so that he can solve a similar problem.
3. Require students to record in writing the problem solution and give a reason for each step.
4. Have students respond to how the answer would be changed or the solution procedures altered by varying certain problem elements.
5. Give the same problem to students but without numbers. Ask them to describe how they would find the answers if they had the necessary numerical information.

Activity: Answer the following questions. Stop to consider the amount and kind of learning that answering the questions has produced. Then select one of the five activities listed above. Carry it out. Then reconsider how much your understanding of the problem has changed.

Grades 6-7 *Blood Flow*

The heart pumps an average of 80 milliliters of blood each second. How many liters of blood does it pump during a day? What kind of container would this fill?



TEACHING CONSIDERATIONS

> "The teacher who holds the development of problem solving skill as an important instructional objective will behave differently than a teacher who does not have a similar goal."
>
> – Bryce B. Hudgins

Deciding to change problem solving from a passive to active goal necessitates reconsideration of related teaching factors. Matters indirectly influencing instruction such as instructional time and evaluation need reexamination. Teaching methods and skills warrant review to identify approaches sensitized to problem solving objectives. The teaching role in large and small group instruction deserves careful reflection.

The following sections discuss factors which merit particular attention in planning for problem solving instruction. Discussion is not intended to be exhaustive. A thoughtful self-assessment will produce other teaching ideas for further consideration.

Instructional Decisions

Teaching Emphasis. In any learning interchange between teacher and students, there are many aspects of the situation that can be highlighted for student attention. We have stressed that when problem solving development is involved, the thinking processes used to resolve the problem situation are more important than the answer, particular method of solution, or computation skills employed.

During instructional time designated for problem solving growth, it is important to refrain from teaching time-outs or side-trips that detract from the problem solving aspects of a situation. This is not to say that mistakes in computation or concept understandings should be overlooked. However, they can be corrected in a manner that will not dilute the problem solving experience.

A student's attention cannot be divided without losing some of the impact of the learning situation. To utilize instructional activities to their fullest potential, the effective teacher will keep instructional emphasis on the goals planned for that experience, carefully noting areas of weak understanding that should become the primary target of another period.

Instructional time. Unfortunately, there are not vast empty time blocks in the curriculum waiting to be filled by increased emphasis on topics such as problem solving. Therefore, the commitment to place more emphasis on problem solving requires making room for it to occur. Part of this time can be found by consolidating certain learning activities. For example, a great deal of computational practice and maintenance activities can occur through careful selection of problem solving situations that require consideration of different combinations of numbers.

However, additional problem solving time may have to be found through better class management activities and more efficient teaching practices. Most curricula can be tightened by attempting to work with fewer examples of new ideas and proceedings and by choosing better learning activities. Single-concept lessons may play an important role in students' understanding of more difficult concepts. However, a well-structured teaching unit with a good introduction of the ideas and relationships to be considered can usually permit a more efficient and more productive merging of some single-concept lessons.

Allocation of instructional time should also give consideration to the unique teaching and learning aspects of problem solving. In many other kinds of instructional situations, the teacher is in the role of an information source, explaining new procedures, identifying new concepts, and providing examples that clarify new material. Students interact with the teacher and each other, but the teacher is the key person guiding the discussion, keeping it on target, and moving it along at the correct pace. In problem solving, the teacher assumes a much less directive role. Once the problem situation is posed, the teacher becomes a group facilitator rather than a group leader. The class is permitted to consider alternatives, to take unpromising approaches, and to become temporarily perplexed. Each of these activities takes considerable time because it is the students, not the teacher, who must do the thinking, make the decisions, and find a successful means to resolve the situation.

Employment of looking back strategies after a solution is reached may take still more time. But problem solving is not an activity to be hurried or to be confined to specific time intervals. To experience problem solving rather than passively to observe it requires considerably more time than other learning activities in which students are told what they should learn.

Planning. The greatest expenditure of teaching time and energy should occur in planning for the classroom activities. Since time requirements limit the number of problem situations that can be addressed, the selection of problems and the planning for how they will be used becomes critically important to effective use of instructional time. In general, using fewer but richer problems that permit many diverse attacks and which can be easily extended are more desirable than several short problems. More often than not, a good problem situation is abandoned too quickly than dealt with overlong.

Preparing for problem solving learning also requires establishing an environment conducive to solving problems. This means that a variety of manipulatives and other tactile materials needs to be available to permit learners the widest possible selection of problem solving alternatives (see materials list, appendix, booklet one). This may require setting the classroom up as a laboratory, or it might involve establishing a materials corner or center to which students have free access.

Some kinds of problem solving experiences should involve individual or small group work rather than whole class instruction. Materials for these experiences need to incorporate a range of difficulties, formats, and related problem solving skills. Although many resources for good problems exist, few individual sources can provide the full needs for a class. Therefore, it is necessary to cull problem sources, selecting activities appropriate for a range of class abilities and arranging them into an organized instructional resource.

Curriculum definition. Providing good problems and designating time for students to work on them is only part of the responsibility for helping students improve their problem solving skills. Well-specified instructional objectives that are part of the total mathematics curriculum are the only means that will assure regular progress and improvement of problem solving capabilities. Within limits, a teacher will accomplish with his pupils what he sets out to accomplish. If his objectives, no matter how important, are general and vaguely defined, he can expect that student growth will be general and hard to identify. But if his goals are specific and well-structured, then he can expect to observe steady and easy-to-identify progress on behalf of his pupils.

Curriculum planning falls into two categories: (1) objectives that directly involve problem solving skills and techniques, and (2) objectives that entail acquisition of other skills upon which problem solving techniques depend. Certain problem solving approaches make use of important *empirical skills* such as data observation, classification, pattern observation, ordering and table construction. Obviously, it is unwise to consider these general problem solving methods before such empirical skills are introduced (though not necessarily mastered).

Curriculum definition should involve a systematic development of problem solving strategies, techniques, and attitudes that are coordinated with related empirical skills. Coupled with this development should be a consideration of child development levels (particularly at lower grades) that introduces strategies and other problem solving skills at operational levels consistent with child growth (e.g. *act it out* is introduced much earlier than *identify a subgoal.)* Sequencing goals and objectives into a curriculum that spans several grade levels will give problem solving the definition it requires to be a teachable and realizable goal. Morever, it will justify the use of 15 percent (an estimate) of instructional time to achieve these objectives and place problem solving learning on a par with computational skill and concept development.

Evaluation. Evaluation methods used for problem solving may have to be more flexible and varied than those used for measuring other instructional objectives. A great deal of research indicates that problem solving behavior is variable from situation to situation and from time to time [Bingham, 1958, p. 13]. An individual's response to a given problem is highly subjective. In many instances, his previous background and set of experiences both within and without school will influence how he perceives a problem and the operating assumptions that he makes.

In assessing problem solving growth, written evaluation instruments are difficult to develop and administer. In order to indicate a valid measure of problem solving, instruments need to contain new problems – **not** slight alterations of those previously studied. Tests should be administered under as little time pressure as is reasonably practicable.

There are also affective dimensions to problem solving growth that should be considered. Problem solving demands risk, will, and confidence on behalf of the solver that belies the belief that a problem can be solved in some way, and that effort, perseverance, and tenacity will successfully prevail over all obstacles. To justify the temporary frustration, anxiety, and discomfort in assailing a problem, the solver must anticipate personal satisfaction, comfort, rewards (extrinsic and intrinsic), and independence. A good evaluation scheme will attempt to identify the depth and breadth of development of these characteristics. Examples of other kinds of behavior that deserve consideration are length of time the individual is willing to continue work before "giving up," receptivity to unusual or alternative approaches, and willingness to continue study of a problem after a solution has been reached.

Because problem solving has so many different facets, every effort should be made to utilize a variety of means to evaluate problem solving performance. No single means is adequate in giving a reliable assessment of the problem solving skill possessed by a student. However, placing more reliance on non-numeric procedures for evaluation may require some changes in teaching style or philosophy.

Classroom modeling. The ultimate goal of problem solving instruction is to develop the capability of students to apply school-learned skills successfully to problems arising in real life. However, experience indicates that transfer of school skills to life situations is not always an easy task. The degree to which classroom activities can realistically model real life problem solving is related to how successfully students can make this transfer. Therefore, an important teacher task is to be aware of real problem solving situations and to simulate them whenever possible.

1. In a real situation, the task to be accomplished or the problem to be resolved is usually well understood. The problem may need to be recast in an appropriate representation, but generally there are no difficulties in interpreting the problem task from given information as is the case in textbook story problems.

2. Real problems frequently deal with tactile materials and/or real people or situations. Being able to see the problem in terms of concrete materials or to associate it with specific persons or situations makes the problem easier to understand.

3. Real problems have a built-in motivation factor. The successful solution of the problem accomplishes something for the solver. He can immediately appreciate his success or be penalized for his failure until the problem is solved. He is therefore willing to wrestle with a problem for a considerable time period. This is in contrast to a school situation wherein a student either moves on from an unsuccessful problem attempt after a brief effort or seeks help from the teacher.

4. There may not be "the correct solution" to a real problem. Many solutions may be acceptable in terms of the specific needs of the solver and the given situation. Rather than requiring an exact answer, it may be sufficient to be "close enough." For this reason, estimation and approximation skills are relatively high in importance in real problem solving.

5. There is no preferred strategy to the successful solution of a real problem. The sole criteria is that the problem be successfully solved in an efficient manner to the problem solver. Therefore, trial and error might be judged just as satisfactory as a careful, reasoned problem attack – if it solves the problem.

6. Real problem situations frequently reoccur for the solver. This gives the problem solver the motivation to find a solution method that will accomplish the repeated task more efficiently. He therefore has a reason to look for more than one solution process to the problem.

7. Real problems are usually cluttered with irrelevant information or may even be missing some necessary data. One of the solver's first tasks is to distinguish between pertinent and nonpertinent information and to decide whether he has enough information to generate a solution.

8. Few real problems are purely mathematical. The problem may consist of many nonmathematical elements. The task of the solver may be to restate and simplify the problem in mathematical terms. A mathematical solution may only be one component to the resolution of a larger problem. Affective considerations frequently play a role in real problem solving. Even though a mathematical solution may point toward one decision, the solver may choose another for reasons that have little to do with logic or reasoning.

It may not be possible or desirable to simulate each of these real life problem characteristics in classroom problem solving. However, an awareness of them should enable the teacher to frequently model such situations, making the transfer between school and life skills an easier task for students.

Teaching Skills

Questioning. In a problem solving dialogue involving a class (or individual) and teacher, questioning by the teacher should be used sparingly and carefully to avoid disclosing information that may reduce the problem to an exercise. Questions should be used to crystallize student thinking (even if incorrect), to help identify obstacles, to encourage wavering confidence, to broaden participation, and to denote progress. Throughout the active problem solving stage (preceding an answer), the teacher should not be "too smart," carrying out actions or following lines of thinking suggested by students. When progress seems to bog down, carefully structured questions can evoke student responses along more promising avenues. However, teacher intervention at this stage should not be too quick!

Wait time. One of the interesting studies involving classroom interactions and teacher questioning disclosed that the average wait time (time passing between a teacher's question and a student response before the question is repeated, rephrased, or redirected) is about one second [Rowe, 1978, p. 207]. Questions were sometimes asked at the rate of ten to twelve questions per minute.

Two facts are apparent in such a rapid questioning format: (1) the teacher is completely in charge of the discussion and expects all responses to be submitted back to him for evaluation; (2) indepth consideration of the question and development of a carefully structured response are not expected nor indeed possible.

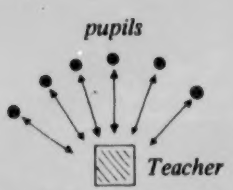
> "[A teacher] must be prepared to meet often with the disconcerting silence of students."
>
> – George Poly[illegible]

A critical aspect of problem solving involves the opportunity to carefully weigh evidence, to compare given conditions and relationships to previously known information or problems, and to search for the untried and unthought. Instead of rapid-fire questioning, the problem solving teacher may even have to advise the class to slow down in their responses and to think them through more carefully.

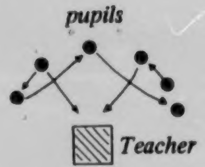
A most significant aspect of the Rowe study involved the change in character of a class when the teacher consciously made an attempt to increase wait time substantially. The following effects were observed when wait times were increased to three seconds or more.

a. The length of student responses increased.
b. The number of unsolicited but appropriate responses increased.
c. Failures to respond decreased.
d. Confidence increased.
e. The incidence of speculative thinking increased.
f. Teacher-centered show-and-tell decreased and child-child comparing increased.
g. Inferences supported with evidence increased.
h. The number of questions asked by children increased, and the number of experiments they proposed increased.
i. Contributions by slow learners increased.
j. Disciplinary moves decreased.

The rapid questioning method represents a communication model in which information is constantly emanating and returning to the teacher from isolated individuals in the class. In a slower paced questioning atmosphere, the communication model becomes one of group participation. The teacher plays more of an equal role with more participation being assumed by individual class members and interaction occurring among students.



Rapid Questioning Model



Relaxed Questioning Model

"The Designated Learner." A teacher-dominated problem solving discussion has other potential weaknesses. There can be a tendency to look to one or two individuals within the class to do the difficult thinking and analysis required to remove the significant obstacles of a problem situation. When the problem is posed, one of these individuals is called upon relatively quickly to expedite solution of the problem situation. When the major obstacle has been removed or a productive attack been suggested for the problem, teacher directed questioning may now involve students who play secondary roles and who can suggest ensuing steps after the crucial steps have been identified.

In actuality, only the one or two central students gain the opportunity to **experience** problem solving. The majority of the class **witness** problem solution by members of their class who have nonverbally been chosen to represent them. Although some insight is achieved in observing another's solution to a puzzling problem, the actual experience of removing the problem obstacle, which is a major component of problem solving, is missed. Therefore, it is doubtful that watchers will have improved their abilities to solve other problems which differ significantly from the original.

Role Modeling. Students need reinforcement that problem solving is more than a single aspect of instruction that receives periodic emphasis. They need to have demonstrated that problem solving is a way of approaching new situations and new challenges. Model teaching can be a very important force in nonverbally depicting desirable behavior for emulation by students.

> "Education does not mean teaching people what they do not know. It means teaching them to behave as they do not behave. It is a painful, continual, and difficult work to be done by watching, by warning, by precept, and by praise, but above all, by example."
>
> – John Ruskin

Being receptive to different ideas and approaches, being alert to unexpected possibilities to reinforce problem solving instruction, using the strategies and techniques of problem solving, displaying a curious and questioning mind, exhibiting an exuberance for and appreciation of problem solving behaviors wherever they are found are practical teaching considerations that will reinforce problem solving instruction throughout the total instructional program.

However, an area which is often overlooked involves the teacher as a true problem solver, rather than as a teacher of problem solving. In most instances, the blockage in student problems will be transparent to the teacher. Therefore, the problem is, in reality, a mere exercise for him. His presentation of a problem to students or his assistance to them as they struggle with a problem will seem to be guided by inspiration and insight. Each move that he makes, each suggestion that he offers, turns out to be a good one. Therefore, students are likely to perceive problem solving as a series of uninterrupted steps of logical thinking that flow from some mystifying insight about the problem which they are unable to achieve. Not realizing that frustration and uneven advances toward solution are, in fact, part of problem solving, the student senses inadequacy and failure when it is not warranted.

To accurately model problem solving, it is therefore necessary for the teacher occasionally to present problems which he has not previously solved, and which he does not immediately see through. This involves an element of risk, but no more so than the risk he is asking a student to undertake for a situation which is a true problem for him. The struggling for solution, the generation of alternatives, the discarding of ideas, the partial failures are all a realistic part of problem solving, including the eventual exhilaration that cannot be suppressed when the problem is eventually solved.

If the problem proves to be a particularly difficult one that does not yield to an immediate assault, the setting aside of the problem for another time, in a conspicuous place, demonstrates another aspect of the accomplished problem solver – the willingness to halt problem efforts temporarily and to return to the problem later with renewed vigor, persistence, and effort until it yields. If in the intervening time, the problem is first solved by a student, then all the better. The stature of students in problem solving endeavors will be enhanced through growing confidence. Class problem solving efforts will be perceived as more coequal endeavors and the teacher as a strong partner rather than always the leader and source of all information.

The Source of Inspiration. Nearly everyone has had the experience of working diligently on a difficult problem which has failed to yield to personal efforts, and later to learn of an inspirational view of the problem which reduces it to a mere exercise. It is very satisfying to understand how the problem is solved and to have the immediate anxiety removed. There is a particularly satisfying feeling that accompanies the understanding of an elegant solution. But a larger question, one very important from a teaching perspective, remains for those who have toiled unsuccessfully with the problem. "How did you ever think of doing that?"

The ultimate goal is to enable each student to solve as many problems himself as possible. The positive effect on a student's problem solving expertise in merely observing the solutions of other problem solvers is nebulous at best. But the teacher can play a primary role in looking back at a solved problem and helping the class understand how the successful student arrived at the crucial idea for solution. Questions posed to the problem solver can be employed for this purpose.

"What is there about the problem which prompted you to think of trying that approach, Betty?" "Was that the first thing you thought of trying, Frank, or did you discard some other ideas first?" "Ann, tell us why you didn't think that some of these other approaches suggested by your classmates would be successful with this problem. Or did you try them first yourself before hitting upon that great idea?" Or to the class the question could be tossed out again, "What if we didn't think of George's super idea? Could we have accomplished the same thing in another way? What else have we left untried?"

Confidence in problem solving comes from two sources. Of primary importance is having the "eureka experience" yourself, of enjoying directly the success of solving problems. Of secondary importance is the recognition that problems are seldom felled by one swift swoop of creative thinking. Solutions are generally the product of many unsuccessful ideas before a key one emerges.

If students understand the full implications of the problem solving process, they will come to appreciate better the techniques and strategies for solving problems that provide them things to try. They will recognize the importance of not only thinking about the problem but of being aware of **how** they are going about that thinking. The importance of consciously expanding alternatives and possibilities, of looking at the problem components in different ways, of seeking creative and untried approaches will be appreciated. Then, in reviewing the successful solutions of others, they will be more sensitive to the need to look past the solution to the thinking underlying that solution. They will become more knowledgeable of the extent of work that precedes the polished and straightforward solutions that appear in textbooks or come from accomplished problem solvers.

The Teaching Role in Large Groups

A substantial portion of instructional time allocated to problem solving is spent in large group instruction. Several suggestions concerning the teaching role in such situations have had informal reference earlier. The following items summarize these key points.

- The teacher selects the problem task for the class as an integral part of the problem solving curriculum keyed to specific goals or objectives.

- Students are led to recognize that they are not seeking reward or approval from the teacher in their problem solving efforts. Their purpose is answer-directed but not answer-motivated. The thinking involved in reaching a solution is the central target.
- Although problem solving "discoveries" can occur within teacher-led (dominated) discussions, this situation is generally less effective than one in which students play a more active role. It is difficult to remove the teacher from the role of final authority and reward dispenser when he is active in the discussion.
- Students should dominate the discussion, examine the alternatives, utilize if-then reasoning, and make judgements on the correctness of ideas. The teacher's role is primarily passive (but **active** behind the scenes).
  - a. present a task or situation in need of resolution;
  - b. moderate the discussion "only as necessary" to keep things moving;
  - c. ask subtopic questions that do not give away the answer but which channel thought along constructive lines;
  - d. insure that participation is spread, that a few do not dominate the discussion or work, that all contribute toward the solution (at least as reviewers and evaluators of the arguments of others rather than being mere spectators or uninvolved observers of the class representatives);
  - e. encourage the class to reorganize in a way to better address the problem or its subparts.
- The teacher must be willing to continue for a time with incorrect results or thinking. He should not make early judgments but let students discover that their conjecture, reasoning, or method of proceeding is wrong. He may or may not wish to be an active agent in bringing the discovery of incorrect matters to light. Whenever possible, students need to be a part of the decision-making and decision-evaluating processes.
- A variety of problem setting modes, situations, and problems is used. Many students will find **true** problem solving somewhat new and frustrating if situations are not geared to their levels of ability and experience.
- The teacher must truly believe that problem solving processes are equally or more important than the applications, facts, or concepts they involve. He must be willing to commit a sufficient amount of time for development and consideration of ideas in using this approach. It is definitely more time-consuming than teacher-led, discussion-recitation approaches.
- The teacher needs to be well prepared in terms of:
  - a. having identified specifically the goal he is striving to achieve;
  - b. having developed a problem of situation which is motivating or interesting in its own right;
  - c. having thought through in advance possible other interpretations or alternatives to those preferred and having planned corresponding questions accordingly;
  - d. having a game plan in hand for assisting students to reach the identified goal whether through activities, resolution of the original problem into subtasks, use of manipulatives, or other teaching methods;
  - e. being flexible enough to abandon his game plan or schedule as events dictate;
  - f. summarizing and expanding on student findings at an appropriate time (perhaps not wishing to do so at the same time that students are carrying the major portion of the learning task);
  - g. utilizing or carrying on thinking or results beyond those anticipated to related problems and conditions;
  - h. evaluating student performances through a variety of measures including personal observation, records of successfully solved problems or significant partial work, use of problem solving techniques, tests and quizzes, and affective growth and personal trait development.

The Teaching Role in Small Groups

Some of the most effective problem solving learning can be accomplished through small group situations in which the teacher may or may not be involved. The role of the teacher in working with small groups and planning for peer involvement needs consideration.

Small Group (or Individuals) and the Teacher

- **Reinforcement.** The teacher provides reinforcement and confidence to small groups of individuals struggling with a problem. The presence of the teacher increases the motivation to solve the problem and establishes the presence of expert assistance nearby. The delicate role of the teacher is to provide the **minimum** amount of assistance to maximize the benefits of the experience for the students.
- **Listener.** An important function the teacher serves is as listener. Requesting that the group or an individual of the group describe the problem situation and the group's efforts toward solution makes use of a teaching and problem solving strategy – state the problem in your

own words. The action of trying to communicate a problem situation to another requires a mental reorganization of problem elements and unsuccessful attempts in such a way that often that act serves as the stimulus for the sudden insight that solves the problem. As problem solvers gain experience, they learn to value the action of describing a problem and their efforts to another as an instrument that may trigger the problem solution (*without any comment coming from the listener!*).

- **Progress delimiter.** Sometimes students need reinforcement that their efforts are making progress though not yet producing an answer. The teacher can act as this progress delimiter. Establishing that certain lines of attack do not produce an answer helps to narrow the field of search and limits remaining alternatives. Organizing what is known about a problem can sometimes point toward alternatives yet untried. Knowing that they have made some progress and established some useful information about a problem can motivate students to continue their problem solving efforts.
- **Clarify obstacles.** A difficulty that sometimes inhibits progress is failure to identify clearly the problem obstacles. Until this action is accomplished, student efforts can be easily misdirected or diffused. Through skillful questioning, a teacher can help to sharpen student focus upon problem obstacles that must be overcome **without giving away any clues** as to how that is to be accomplished.
- **Offer challenges.** A distinct advantage of small group work is that ability differences can be greatly reduced so that problems can be more properly targeted to the specific capabilities of the individuals of a small group. Working with a small group, the teacher may observe the successful solution of a given problem and recognize that group problem solving capabilities are sufficient to be challenged by an extension of the original problem to a more general case or to one involving more problem components. The teacher can make on-the-spot adjustments in learning activities or suggest challenges that may carry on past class time.
- **Direct attention to problem features.** The opportunity to direct attention to the solution process rather than answer aspects of a problem are also available in small group work. The teacher may emphasize and reinforce the problem solving processes, starting points in the problem, significant problem features, strategies employed, or any summarizing point about the problem. Taking advantage of the small group, teaching can be more efficiently directed at specific problem solving weaknesses of individual group members.
- **Seize problem solving opportunities.** The opportunity to model and reinforce problem solving thinking arises in many kinds of learning situations, whether designed specifically for problem solving instruction or not. The skillful teacher is sensitive to any situation in which a problem solving teaching style can be used with learners and takes advantage of these opportunities. Computation, concept development, and application situations may be used as problem solving situations. A teaching style that uses inquiry teaching can turn such learning activities into problem solving experiences.

Use of Peer Learning

Grouping students together to solve problems has advantages and disadvantages. By recognizing the strengths and weaknesses of such instruction, the teacher can make these experiences more productive.

Advantages

- Students have the opportunity to learn about group problem solving and dynamics within a learning group.
- Opportunities for problem solving success are enhanced as compared to individual efforts.
- Motivation to stick with a problem may be increased.
- Students are more likely to experience and witness realistic problem solving, false starts, uneven problem progress, frustration.
- Thinking by the group can be broader, more creative, and more insightful than individual efforts.
- Interaction with others may stimulate additional problems, insights, and discoveries.
- Students are exposed to a variety of thinking and problem solving styles different from their own.
- Students can motivate one another to excel and to accept more challenging problems.
- More opportunities for unexpected payoffs can occur as groups or subgroups go in unforeseen, unpredicted, and separate directions.

Disadvantages

- One or two students can dominate a group and do all of the actual problem solving.
- Until group problem solving processes are understood by group members, fruitful ideas and approaches can be discarded for superficial or incorrect reasons.

- Frustration and nonsuccess with problems can develop into disruptive behaviors.
- Some students' learning styles may be more attuned to doing reflective thinking in a quieter atmosphere.
- Opportunities for immediate assessment of understanding through feedback are lessened.
- Incorrect or undesirable problem solving behaviors may be modeled by group members.
- Looking back problem solving actions are difficult to accomplish.

Advance preparation in establishing problem solving groups and preparing students for such instruction can increase their effectiveness.

Suggestions

- Provide students with instruction about group problem solving, procedures and pitfalls.
- Encourage more capable class members to "let others in on the action" instead of relegating them to spectator roles.
- Provide measures that reward more insightful behaviors such as finding more than one solution, creative solutions, generalizing, satisfying or practical solutions, extensions of the problem.
- Make certain that rewards accrue for group problem solving efforts to a degree commensurate with individual problem solving: praise, recognition, encouragement, point accumulations.
- Encourage students to describe their thinking in the early stages of a problem before a solution path is evident.
- Observe group problem solving behaviors so that you can intervene as necessary to provide needed assistance or restructuring of the group/peer situation.
- Require solutions to be written out by the group so that (1) there is a record of the answer process, (2) the group must reflect upon, review, and reexamine the problem in perspective in order to record the result and process, and (3) skill is developed in organizing and writing up an account of the process.
- Provide sufficient directions to aides or other persons assisting students in problem solving to assure that their assistance is directed toward the desired outcomes.
- Review materials such as filmstrips, videocassettes, or printed material that students are to use for instruction and construct specific objectives for and means of evaluation of the use of these items.

Effect of Problem Solving Commitment

Communication between teacher and students about instructional objectives occurs on several levels. There are those topics which students are told are important. There are subjects which are emphasized by the teacher in instruction. There are matters which consume large amounts of instructional time. There are areas on which students are tested or otherwise evaluated.

If the importance of problem solving is verbalized but teaching and grading emphasis is placed upon accuracy of computations, students will identify computational proficiency as their target. If problem solving techniques are discussed periodically in class but are not modeled in other classroom activities, students are not likely to extend problem solving behaviors to those activities either.

When instructional time, teaching emphasis, evaluation activities, and objectives all consistently support problem solving learning, students will identify it as an area of importance for mastery. When the teacher models problem solving behaviors for all areas of instruction, students will begin to emulate these actions and extend problem solving concepts to other instructional topics and extra-school activities. When a teaching environment is created that stresses thinking about problems and encourages and values alternative and creative approaches to problem tasks, students will begin to perceive problem solving as a more important activity than an answer search.

Problem solving is more than a method of teaching. It is a general attitude, a disposition toward inquiry which has as goals the discovery of new ideas and relationships based upon existing ones. It involves receptivity and appreciation for different views and approaches. It relishes the unique excitement and pleasure of mastering a difficult situation, of transforming a complex and puzzling problem into one of understanding and insight. But the price for the "aha moment" is careful planning, curriculum development, teaching commitment, and hard work.

CHOOSING DIFFERENT TEACHING MODES

> "Variety is the mother of Enjoyment"
>
> – Benjamin Disraeli

Varying the manner in which problems are presented or in which they are permitted to arise adds to the enjoyment of problem solving and improves capabilities to resolve problems under those circumstances. It also gives the teacher the opportunity to observe the strengths and weaknesses of individual students for particular problem solving modes of instruction.

Story Problems

A written problem presenting the setting, givens, and problem task is the most common problem solving format that students experience in structured learning situations. Ability to translate a problem from written form into symbolic notation and working concepts is a very important skill for success in school work and in some real life situations. There are, however, serious questions about its exclusive use as a problem solving mode.

Does the student have difficulty with comprehension, word meanings, and sentence structure or does he have difficulty with use of the ideas and concepts required to solve the problem? How realistic are story problems to practical, real-life problems?

In a practical situation, the problem task is usually well-defined or understood without the need for reading. Reading may or may not be required to solve the problem. When it is important to the process, the contribution of reading will frequently be to provide information or data that the problem solver needs to carry out his plan of attack.

Preliminary Discussion

One way to remove the comprehension difficulty from story problems is to read through such problems as a class, asking individual class members to restate the problem in their own words and to identify the specific problem task they are asked to accomplish. This activity can also be an effective means of illustrating certain problem solving strategies; e.g. relating the given problem to a similar or simpler situation as a clue as to how to begin. It should also direct students away from over-reliance upon word cues and focus their attention more specifically upon problem components and inter-relationships of given, wanted, and needed information.

Oral presentation of a problem can also be used to pose the complete problem when no other information in written form is presented. Student feedback is usually immediate, and continues until they feel comfortable enough about their understanding of the problem task to proceed.

Picture Problems

Another problem format which is gaining in popularity in some commercial texts is to pose a problem through a picture or series of pictures, sometimes with minimal directions attached. The following problem is representative of such a format.

Grades 5 & 6

WHICH IS BETTER?

$ 7 for 12

$22 for 36

Understanding of the problem task is usually immediate and interest is stimulated by seeing pictures of the kinds of objects the problem involves.

Manipulatives

Use of concrete objects to pose a problem will sometimes stimulate good performances from students who are not as successful with written problems. The opportunity to manipulate and interact with problem materials permits some students to better visualize problem relationships, to test out their ideas, and to perceive extensions of the original problem to other situations. They may not be capable of these activities when presented the same problem in a different form. Conversely, other students who perform better with other problem modes may not do as well or be as receptive to physical objects. This reinforces the need for variety in learning experiences. Students can grow in the use of new techniques as well as excel with formats closer to their unique learning styles.

Games

Some kinds of educational games require the use of sophisticated strategies, reasoning, or other complex thinking to perform the game activity successfully. Using discretion and good judgment, a creative teacher can turn such situations into good learning experiences.

Grades 3-9 *Pico - Fermi - Zilch*

1. *One person selects, but does not disclose, a two-digit number.*
2. *A second person attempts to guess the selected number in the least number of guesses possible.*
3. *In response to each guess, the first person chooses one of three words:*
    a. *Pico - means your number has a digit in common with mine but in the wrong position. (Example: selected number is 54 and response is 47.)*
    b. *Fermi - means your number has a digit in common with mine and in the correct position.*
    c. *Zilch - means your number does not contain a digit in common with mine.*

In making decisions as to what guess to choose next, some excellent problem solving thinking can be exercised.

Many similar kinds of games which utilize problem solving thinking are available commercially. Games can also serve as the background against which a problem can be posed. Certain conditions can be described and the solver challenged with making the best decision as to how to proceed or the game strategy to employ.

Personalized Situations

One of the most intriguing problem presentations poses problems requiring a different response from each solver according to something unique about him. Sometimes solver results will differ, but sometimes they may be the same!

Grades 6-7 *Billionth Second*

Determine on what future day you will pass your billionth second to be alive.

Answers will vary but be within a range for a class. Comparison and debating of results can make this problem a rich problem solving activity.

Grades 8-9 *Two-Digit Madness*

a. *Choose any two-digit number.*
b. *Reverse digits and multiply by the original number.*
c. *Square each digit of the original number, add, and multiply by ten.*
d. *Subtract this product from the result in (b).*
e. *Your answer will be a four-digit number (first digit might be zero) consisting of the repeated product of the digits of your given number (e.g. 4545).*

Why does this happen?

A more lengthy problem of this same type produces the same result no matter what first choice is made.

Grades 8-9 *Ubiquitous 58*

Select any whole number. Find the sum of the squares of its digits. Repeat until you reach 58 or 1.

$$\begin{array}{rlr} 651: & 6^2 + 5^2 + 1^2 & = 62 \\ 62: & 6^2 + 2^2 & = 40 \\ 40: & 4^2 + 0^2 & = 16 \\ 16: & 1^2 + 6^2 & = 37 \\ 37: & 3^2 + 7^2 & = 58 \end{array}$$

Explain why this happens.

For such an intriguing outcome, it is hardly necessary to motivate students to ask why the situation occurs.

Real World Situations

Around any school, there are numerous real problems that need to be solved. Involving a class in the solution of such situations makes a nice change of pace and graphically demonstrates the application of school-learned skills to the real world.

Problems selected for this purpose will most likely be longer, more complex, and less routine than "school work" problems. Analyzing the problem situation as a class, deciding upon a plan of action, breaking that plan into sub-problems, gathering information, interpreting results, and reaching a conclusion can be an effective learning activity that may cut across several disciplines while offering a unique kind of problem solving experience.

Grades 6-7 *Pedestrian Crossing*

Select a dangerous pedestrian crossing near the school. Study how it could be made safer. Prepare a list of alternative remedies with supporting arguments listing advantages and disadvantages. Assume the arguments are to be presented to city council or township officials to persuade them to make the crossing safer.

Guided Discovery

For problems whose results can be generalized through inductive thinking, guided discovery offers a problem solving format with unique opportunities. Students are presented specific easy problems of the same type and asked to record their results in a prescribed manner. In answering carefully designed questions, students can be led to discover patterns that enable them to generalize their findings to other cases not yet considered.

More capable students may leap quickly from the specific cases to the general relationships involved. Slower students may require the step-by-step consideration of each specific case and careful deliberation in answering the key questions.

Grades 6-7 *The Difference is in the Squares*

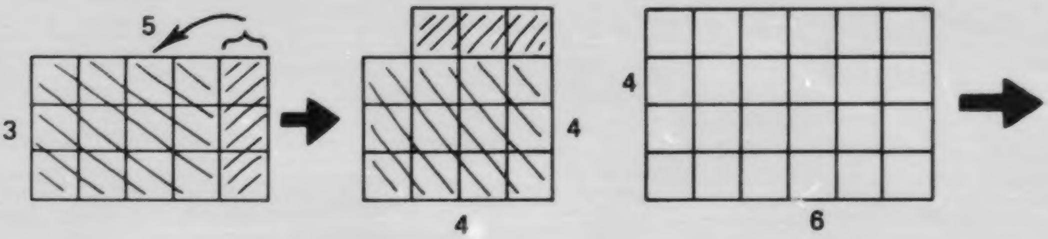
I. Complete these equations and look for a pattern.

7 x 7 =	*8 x 8 =*	*9 x 9 =*	*11 x 11 =*
6 x 8 = _____	*7 x 9 =* _____	*8 x 10 =* _____	*10 x 12 =* _____
Difference =	*Difference =*	*Difference =*	*Difference =*

II. Which is more? 19 x 21 or 20 x 20? By how much?

III. If you know *38 x 38 = 1444* *Can you find* *59 x 61 =*
can you find? *37 x 39 =* *59 x 61 quickly?*

IV. Study the first diagram and then complete the second:



$3 \times 5 = (4 - 1) \times (4 + 1)$ $\qquad 4^2 - 1 \qquad$ $4 \times 6 = \_\_\_\_^2 - \_\_\_\_$

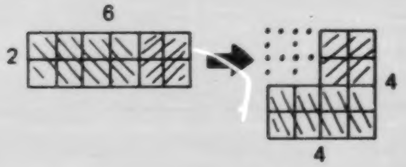
V. Complete these equations and look for a pattern.

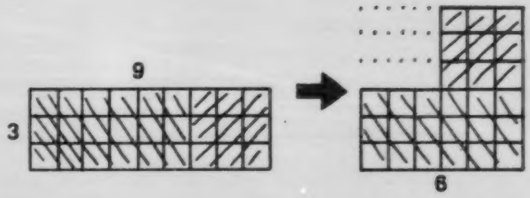
$6 \times 6 =$	$9 \times 9 =$	$12 \times 12 =$	$20 \times 20 =$
$4 \times 8 =$ ____	$7 \times 11 =$ ____	$10 \times 14 =$ ____	$17 \times 23 =$ ____
Difference =	*Difference =*	*Difference =*	*Difference =*

VI. If you know $17 \times 17 = 289$ *can you find?* $15 \times 19 =$

If you know $25 \times 25 = 625$ *can you find?* $22 \times 28 =$

VII. Complete the equations that describe these diagrams.





$2 \times 6 = \quad \_\_\_\_^2 - \_\_\_\_^2 \qquad (6-3) \times (6+3) = \quad \_\_\_\_^2 - \_\_\_\_^2$

Spontaneous

Some of the best problems are still those which occur spontaneously, either student-originated or prompted by something which comes up during a class discussion. "I wonder what happens if . . ." can be the beginning of an exciting discovery for teacher and students alike. Mathematical discoveries, like those from other fields, frequently occur because someone is curious and perceptive. A conducive classroom atmosphere coupled with interest and receptivity to pursuing interesting questions will make spontaneous problem solving a frequent activity, sometimes involving the whole class and other times engaging the interest of a small group of students.

Problem solving can be varied in so many ways – problem types, problem difficulties, problem settings, problem modes of presentation. A class which takes advantage of a rich and varied problem solving strand will be an exciting and illuminating place that students will look forward to each day.

* * *

STORY PROBLEMS AND NUMBER SENTENCES

Sure-Fire Rules for Solving Story Problems

To be applied in order:

1. Most problems are addition. If there is any doubt, add.
2. If more than two numbers are given, it has to be addition.
3. When only two numbers are given and they are about the same, subtract.
4. Consider subtraction when money is involved, particularly if one amount is a round figure like 50¢ or $10.00.
5. If two numbers are given and one is much larger than the other, try division.
6. Very few story problems involve division with a remainder. When you get a remainder, cross out the division work and multiply instead.

Although we may find this list of "rules" to be humorous, like most humor, the statements do strike closely to the truth. Young children are apt to over-generalize when confronted with the task of solving story problems. And textbook problems probably fit the circumstances described above too frequently.

The challenge of helping young children to be able to translate a story problem into a number sentence and to determine an operation which will produce the answer to the question can be a trying teaching task. Story problems bring together four components: (1) imbedding of an arithmetic problem into a social setting; (2) generalizing of the situation to identify common elements that key the child as to how to handle the problem; (3) discriminating to discern how this problem may differ from previous generalized approaches; and (4) understanding of the meaning of the operation required in the solution. Successfully handling these components of story problem instruction requires insight into the instructional aspects of the situation and into the nature of how young minds perceive such situations.

Consider the following problems. Each requires writing a number sentence prior to answering the question.

J-1 Jenny had saved 43¢. Her mother gave her 25¢ more for helping do the dishes. How much money did Jenny have then?

S-1 Brad has been collecting stamps. He had 83. For his birthday, Brad's parents gave him some more stamps that he did not have. Now Brad has 101 stamps. How many stamps did Brad get for his birthday?

J-2 Barbara began painting in her new paint book. She has painted 24 pictures so far. Barbara counted 18 pictures that are left. How many pictures does the book have?

S-2 Joe had 63 marbles. He gave 14 to Maureen. How many marbles does Joe have left?

A ranking of these problems according to difficulty would proably indicate that the first and fourth problems are somewhat easier than the second and third. Each problem contains three elements that influence a child's thinking about the problem. Joining or separating is indicated, cue words or phrases are given, and a number sentence involving an operation is to be written.

J-1 is a problem involving joining. It includes the key phrase "gave her 25¢ more." The associated physical action number sentence 43 + 25 = N matches the operation required to produce the answer. All three informational components consistently point toward the same means of solution.

S-1 requires a separating action but uses the cue words "gave him some more stamps." Its associated physical action number sentence 83 + N = 101 does not use the operation required to produce the answer. The information sources are not supportive.

J-2 represents a joining action but involves the cue phrase "18 pictures that are left." The physical action number sentence is N - 24 = 18, but addition is needed to produce the answer. Sources of information do not consistently indicate the same action required of the child.

S-2 involves a separating action. Its key phrase "does Joe have left" matches the associated physical action number sentence 63 - 14 = N. All three information sources suggest to the child subtraction as the means to find the answer.

The manner in which problem information is stated may therefore help or hinder the child's thinking of the operation necessary to produce the required answer. The matter is further complicated by lack of uniformity in the way that children relate their number sentences to the format of the problem. We have described a physical action number sentence for each problem above. However, other views are correct and logical. A number sentence can be written that corresponds to the joining or separating action of the problem. Or a child may write a number sentence linked to **undoing** the problem action.

Consider problems two and three according to each of these problem views.

	Problem View	Number Sentence	Child's Thinking
	physical action:	83 + N = 101	83 stamps, received some more, had 101 stamps at the end.
S-1	operation required:	101 - 83 = N	101 stamps after birthday, 83 before birthday, difference is number received for birthday.
	undoing:	100 - N = 83	Had 101 stamps at the end, take away the number he received, and that gives 83 which he started with.
	physical action:	N - 24 = 18	From the total pages of the book, 24 are painted, leaving 18.
J-2	operation required:	24 + 18 = N	24 pictures painted, plus 18 pages left equals the total number of pages of the book.
	undoing:	N - 18 = 24	18 pages unpainted from the total gives 24, the number which are finished.

Consequently, children may choose to associate different but equivalent number sentences to the same problem. Individual differences in their thinking traits may cause them to prefer one means of considering a problem over others.

In early instruction, some children may grow confused and alternate between methods because they do not fully understand the interrelationships of the number sentence, problem elements, and operations. They need to realize that separating and joining are operations involving sets of objects or things, not numbers. The question posed by a problem may or may not correspond to using numbers to describe how the sets of objects are manipulated. Cue words or phrases within the problem are not dependable for deciding the operation required for finding an answer. And the operation appearing in the number sentence may not be the one used to produce the answer.

Teaching flexibility in presentation and sensitivity to individual differences therefore become very important. Teaching explanations based upon a point of view different from a student's own may be confusing and illogical. But it is equally undesirable to present three views of every problem. As a compromise, a teacher may choose to utilize a presentation which most children find satisfactory while reinforcing individual children with asides such as, "And how would you write that number sentence, Joy?" The important goal is to strive to achieve with each child a perceptual understanding of the relationship of the number sentence to the story problem rather than to develop a mechanical procedure that depends upon nonessential aspects of the problem format or phrasing.

One approach to story problems that has met with documented success, particularly with students of lower ability, has been the wanted-given approach [Wilson, 1972, p. 79]. Such an approach depends upon definitions of the four arithmetic operations which permit their use in a wanted-given analysis of a story problem. For example, "addition is an operation used to find the 'size of the total' when we know the 'size of each part.' Subtraction is used to find the 'size of one part' when we know the 'size of the total' and one of its 'parts'."

Giving students practice with a variety of subcategories within one type of wanted-given classification can help them to perceive the interrelationships between the parts of the problem. Many textbooks do not provide this kind of variety. Therefore, it is important for the teacher to supplement textual materials with appropriate variation of formats, helping students refrain from errors such as overgeneralization.

Classification and Subcategories by the Wanted-Given Model (adapted from materials by Wilson, p. 80)

ADDITION. Wanted: sum. Given: addends.

TO FIND	KNOWN	EXAMPLE
1. N of the whole	N of both parts	Janet had 7 tennis balls. Her brother gave her 4 more. How many does she have now?
2. N of initial whole	N of part removed and N of part left	Janet had some tennis balls. She gave 4 to Ken. Janet then had 3 left. How many balls did she have to begin with?
3. N of the larger part	N of the smaller part and N of the difference	Janet had 4 tennis balls. Ken had 3 more tennis balls than Janet. How many did Ken have?
4. N of the whole	N of each part (no action)	Janet has 4 white tennis balls and 3 yellow tennis balls. How many tennis balls does she have in all?

SUBTRACTION Wanted: addend. Given: sum and one addend.

1. N of part left	N of whole part and N of part removed	Janet had 7 tennis balls. She gave 3 to Ken. How many did Janet have left?
2. N of part added	N of initial part and N of the whole	Janet had 4 tennis balls. Her brother gave her some more. Janet now has 7 balls. How many balls did her brother give her?
3. N of part needed	N of part present and N of whole sought	Janet has 3 tennis balls. She needs 7 balls for her game. How many more balls must she get?
4. N of the difference	N of both parts (discrete parts)	Janet has 7 tennis balls. Ken has 4 tennis balls. How many more (fewer) does Janet have than Ken?
5. N of initial part	N of part added and N of the whole	Janet had some tennis balls. After she received 3 more from her brother, she had 7. How many balls did Janet start with?
6. N of part removed	N of part left and N of the whole	Janet had 7 tennis balls. She gave some to Ken. Then Janet had 3 left. How many did she give to Ken?
7. N of part removed	N of part and N of the whole set	Janet had 7 tennis balls. 3 of the balls were white. The others were yellow. How many of the tennis balls were yellow?
8. N of the smaller part	N of the greater part and N of the difference	Janet has 7 tennis balls. Ken has 3 fewer balls than Janet. How many balls does Ken have?

MULTIPLICATION Wanted: product. Given: factors.

1. N of sum of parts	N of groups and N of parts per group	Brenda has 3 boxes of crayons. There are 12 crayons in each box. How many crayons does Brenda have?
2. N of comparison group	N of group and ratio of that group to unknown group	Brenda has 3 boxes of crayons. She has 12 times as many crayons as boxes. How many crayons does she have?
3. N of Cartesian product	N of each group	Brenda has 4 blouses and 3 skirts. How many outfits does she have?
4. N of sum of parts	N of groups and N of parts per group	Brenda put her crayons into 3 boxes. She put 12 crayons into each box. How many crayons does Brenda have?

DIVISION Wanted: factor. Given: product and one factor.

1. N of groups	N of total parts and N of parts per group	Brenda had 36 crayons. She put 12 crayons in each box. How many boxes did she use?
2. N in each group	N of total parts and N of groups	Brenda had 36 crayons. She put the same number of crayons into each of 3 boxes. How many crayons did she put into each box?
3. N of groups	N of parts per group and N of resulting total parts	Brenda placed 12 crayons into each box. She had counted 36 crayons as she put them away. How many boxes did she use?
4. N in each group	N of group and ratio of that group to unknown group	Brenda has 36 crayons. She has 12 times as many crayons as boxes. How many boxes does she have?

It is not important for students to recognize these classification subcategories of the wanted given model but rather to have experience with a variety of problems chosen from each operation class.

The ability to write and solve number sentences for story problems is one of the most useful mathematical skills that children can acquire. Its importance warrants the time, planning, and careful attention necessary for its development.

GETTING STARTED

Better Use of the Textbook

Deciding to place more emphasis on problem solving experiences for your students involves giving consideration to many factors. Adjusting the teaching schedule, rewriting the curriculum, selection of materials, assessment of student strengths and weaknesses are some of the larger tasks that you face to do a comprehensive job. But most of us need to begin on a smaller scale and to experiment for a time. We need to learn what works well for us and those things which do not work. This period of time permits you to assess your own capabilities, resources, and restrictions while convincing yourself that a more comprehensive effort is worthwhile.

The question is where to start. Since the textbook is likely a major source around which your classroom activities are planned, it is the logical candidate for being used more effectively. Problem solving is a component in most textbook series. However, it is probably not as strong a strand as that we are suggesting. Therefore, you may be able to strengthen your planned problem solving experiences with children through some of the following suggestions.

1. **Identify precisely your objectives for using problem materials.** Use of text materials can serve differing objectives: application, reinforcement, computation skill practice, problem solving technique development, recreation, puzzles, logical reasoning practice, confidence building, challenge, time filler. Identifying the specific outcome that is to be accomplished through a particular set of materials will enhance its use with students.

2. **Regroup problem materials to suit your purposes.** Most textbooks organize problems to provide practice with specific kinds of content or processes. Students may soon realize that an entire set of word problems will involve a particular operation or application. Therefore, problems become thinly disguised exercises in which most students merely select numerical data for some standard operation. By mixing problems from more than one section, students will be exposed to more realistic problem solving analyses and decisions.

3. **Avoid leaving specific problem situations too quickly.** Good problem situations possess the potential for discussing a wealth of related ideas that will not surface if answer-getting is the primary objective for using a problem. A good problem is like a multi-faceted jewel. It deserves careful examination from many perspectives. In general, problem solving discussions with large student groups are strengthened by considering fewer problems discussed in more depth than a large sample of problems treated quickly.

4. **Make better use of challenge problems for the total group.** Textbooks frequently include unusual, slightly more difficult, and nonroutine questions at the end of sections or at the bottom of certain pages. These are sometimes labeled "Extra for Experts" or identified by some similar title. Because they are found at the end of materials or carry labels suggesting that they are more difficult, these problems are sometimes passed over or used only with more capable students. If discussed with an entire class, such problems may pose a challenge to the more capable students. They are quickest to respond with correct views of the problems, thereby ending the problem solving experience for more deliberate students.

Such problems usually have the potential for being discussed in more detail through such avenues as: seeking different means for solution, generalizing, extension to other related situations, and other analysis kinds of activities that can involve a whole class. Alternatively, a teacher may use a response system that enables all students to address the problem for whatever amount of time they require for its solution. This might be accomplished by having students respond in writing as they solve the challenge problem.

5. **Discuss with students the purposes of problem materials.** Some students have negative attitudes toward written problems or any kind of problem solving activities that ask them to engage in detailed thinking. Such attitudes present opportunities to conduct indepth discussions with your class about the reasons for their feelings, about the role that such work plays in their learning, about how such materials or experiences can be improved. Having students critically examine their learning materials, critiquing and rewriting questions and activities can prompt a greater awareness of the structure and purpose for such work. It can generate a more insightful awareness of those elements of a problem which are critical and those which are dressing. It can

foster a more wholesome attitude about learning experiences that involve this kind of material. Students will gain a more accurate understanding of your expectations of them as well as appreciating how problem solving relates to their learning.

6. **Use text materials as models for classroom development of questions.** As you develop your own problem solving materials, you can benefit by using textbook problem formats, organization, and ways of posing questions. Moreover, students can grow in understanding about given, needed, and requested information in problems by having the task of constructing problems themselves. The textbook may serve as a model for them. By observing how their questions differ in style and format, you can obtain useful information about students' thinking and how they perceive such problem work.

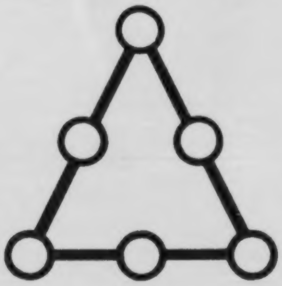
7. **Extend textbook problem situations through self-developed materials.** Not only can the textbook serve as a model. Some of its materials can be used directly in constructing your own problem situations. Some problems can be extended from their present form to additional considerations. Other materials, particularly graphical and tabular data, can be transformed from informational, exercise work to genuine problem situations with a sprinkle of your creativity.

8. **Examine the book throughout for problems that can be used in teaching.** Problems need not be used in the order in which they are presented in the book. Situations that might be exercises when introduced after related content can be good vehicles for problem solving investigation when presented earlier, before the acquisition of related ideas. By using a problem solving methodology, you can motivate a need to study additional topics. Selecting problems which require the skills to be acquired from the following sections of the textbook can be helpful and timesaving to you in preparing your teaching plans.

Using a Text Problem Fully

The issue of making better teaching use of fewer problems is so important that we would like to provide an example of such teaching with a specific problem.

Grades 2-3



Fill in the numbers 1, 2, 3, 4, 5, 6. Each row of circles must have a sum of 9.

Illustrating different strategies:

I. **Make a table** of possible combinations.

6	5	5	4	4	3	3	3
1	1	2	1	2	1	2	3
2	3	2	4	3	5	4	3

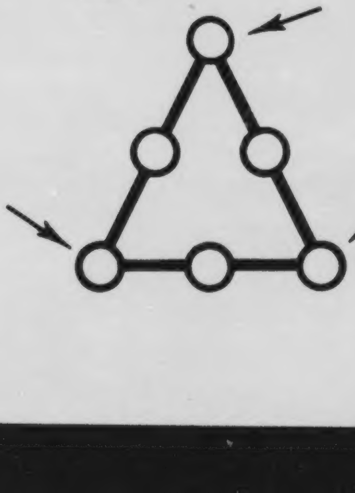
Eliminate impossibilities and duplications.

6	5	~~5~~	~~4~~	4	~~3~~	~~3~~	~~3~~
1	1	~~2~~	~~1~~	2	~~1~~	~~2~~	~~3~~
2	3	~~2~~	~~4~~	3	~~5~~	~~4~~	~~3~~

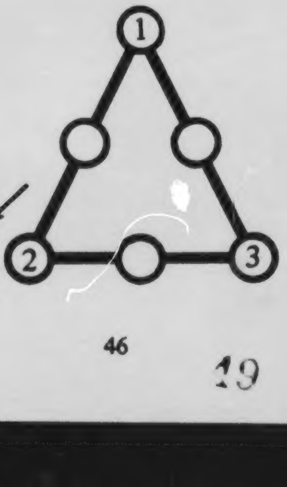
Note numbers that are used more than once.

6	5	4
①	①	②
②	③	③

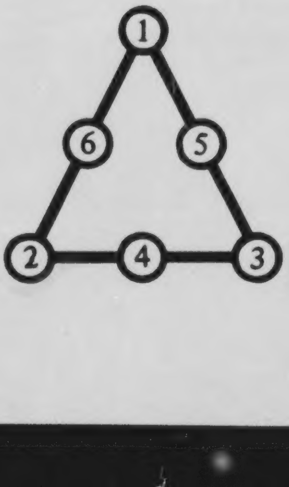
Observe the triangle. Where are *numbers* used twice?



Fill in the numbers 1, 2, 3.



Complete the diagram.



II. *Work with a simpler problem:* what two numbers can be in the same row? →

1 + 2 = 3 yes
2 + 3 = 4 yes
2 + 3 = 5 yes

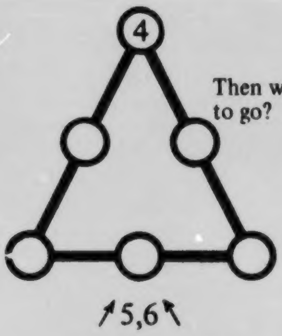
jump to

6 + 5 = 11 no
6 + 4 = 10 no →
5 + 4 = 9 no! why?

Others are okay.

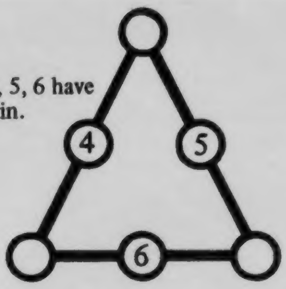
Conclusion: 6, 5, and 4 cannot be in the same row. What does this teach us about the figure?

Can, 4 say, be in the top circle? Could 5 or 6 be in the same row with 4? Where would 5 or 6 have to be if 4 is in the top circle?

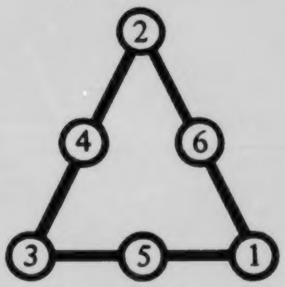


Can both go there? Is the top circle any different than any other "corner circle"?

Then where do 4, 5, 6 have to go? Fill them in.



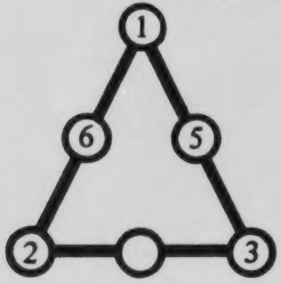
Complete the diagram. (Some guess and check will now have to be employed.)



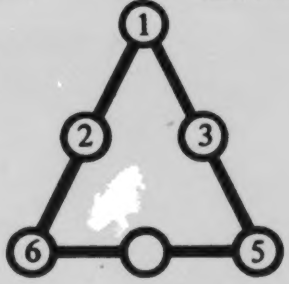
III. *Guess and check.* Using previous knowledge about averages, I know that I must get three groups of numbers to add up to nine. So the larger numbers will probably go with the smaller numbers and the medium-sized numbers will go together.

Let's try 6-2-1 and 5 with 2 and 2 . . . No, 5 will have to be with 1 also, so 5-1-3. That makes 1 used twice. So it has to go in a corner where a number is used in two rows.

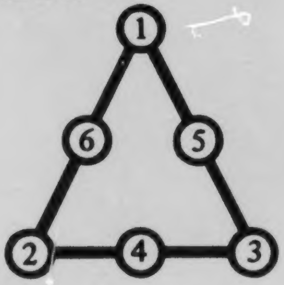
That means we have two possibilities . . . maybe others . . .



or

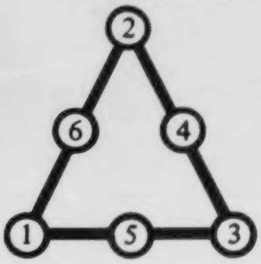


Say . . . 6 and 5 can't be in the same row so it has to be:

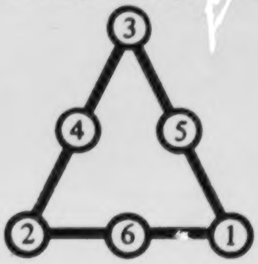


Using "Looking Back" Teaching

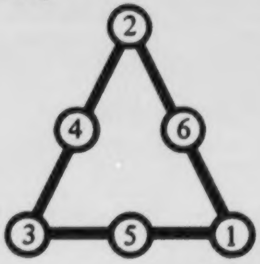
Suppose the problem has been solved by students through guess and check. Three volunteers have put their solutions on the board alongside the original diagram. The teacher now focuses class attention on these solutions through questioning.



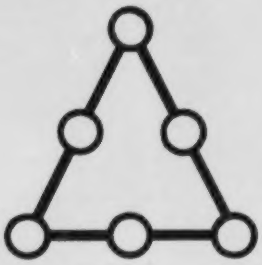
Ann



Bill



Carla



Original

"Look at Ann and Bill's answers. Are they different or the same? How are they different? How are they the same? How can you make them the same?" (rotation)

"Look at Bill and Carla's answers. Are they different or the same? (clockwise and counterclockwise) "How are they the same? How can you make them the same?" (flip and rotation)

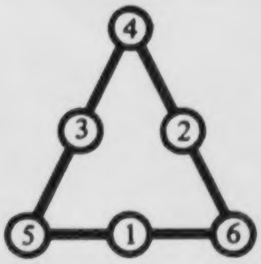
"Are Ann's and Carla's solutions 'really the same' or 'different'? (mirror reflections) If you know that you can make Ann and Bill's answers look the same, and you can make Bill's and Carla's look the same, does this tell you anything about Ann's and Carla's? What?" (transitivity)

"Now look at the original diagram. Are all the circles the same for this problem? How are they different? . . . So some numbers will be used twice and some will be used once. Did the problem give you any hints about what numbers might be used twice?"

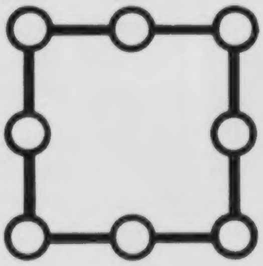
"What might keep a number from being used twice in this problem? If you had thought about these questions before solving the problem, do you think it would have made it any easier? Why?"

Extending the Problem

"When I attempted to solve the problem, one of my false attempts was



I noticed that all the rows summed to 12. Do you think there are other solutions with sums of 10 and 11?"



"Can you put to work any ideas that you learned about the triangle problem to fill in the numbers 1-8 in this square diagram? Sums are to be 12."

An added feature of each of these extensions is that it affords the opportunity for skill maintenance activities while testing problem solving conjectures. Students will apply numerous computation skills, both mental and written, coupled with careful reasoning before producing the required solution.

The point is not to use each of these extensions with the same problem. That might be too time-consuming. However, each extension represents the use of mathematical ideas and thinking that are worthwhile and which grow out of the original problem. By not being answer-oriented, we can pause to consider good problem situations to assure ourselves that we have employed them to the fullest potential desired.

Acquiring Good Problems

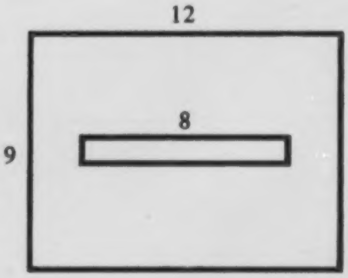
As you begin to increase the number of problem solving activities that you use in teaching and learning situations, you will recognize the need to acquire additional good problems beyond those available in the textbook. The emphasis here is upon **good** problems. Use of poor or weak problem solving activities can be as discouraging to student problem solving growth as rich and stimulating problems can be productive.

Good problems should produce excitement and sense of accomplishment in their completion. They should give the learner a sense of satisfaction that they were worthy of his effort. Properly chosen, good problems are within the problem solver's range of accomplishment but still entail a sense of discovery and of working with mathematical ideas in an original, personal way. Aspects of mathematical structure are revealed or utilized through many good problems.

A well-constructed problem situation will reward, not penalize, logical thinking, organized effort, problem solving techniques, and careful reasoning. Good problems offer variety in the formats in which they appear, the applications they address, and the solution thinking they require. A good problem will not require a trick or highly improbable perception for solution (unless the solver is somehow alerted to that possibility). Many so-called puzzles or trick questions exhibit this property. It distinguishes them from good problems. The following two examples are not considered good problems.

(a) Continue the following sequence: O, T, T, F, F, S, S, E, . . .

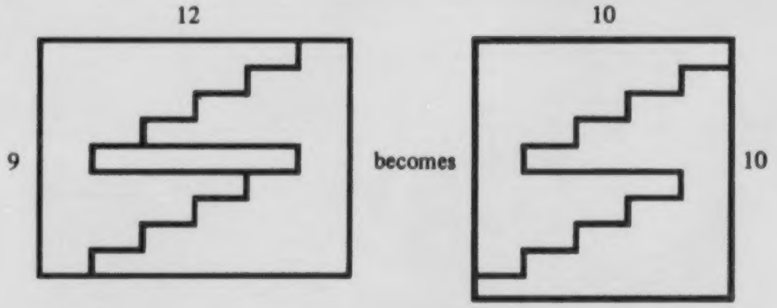
(b) A 9 x 12 rug has become worn in the middle so that a 1 x 8 section has had to be removed from the middle as shown. How can you cut this rug into exactly two pieces that can be fitted together to make a 10 x 10 rug?



Answers to these two puzzle questions are:

(a) N, T. The letters of the sequence are the first letters of the words: one, two, three, four, five, . . .

(b)



Such questions can be discouraging and self-defeating. The solutions to such questions have little relevance to other situations and produce little, if any, satisfaction for the solver.

Good Problems

- entail concepts, content, or processes in the solution process or results that are worth consideration.
- cause the learner to synthesize what he has learned.
- involve obstructions to solution which are manageable and within reach of the solver while still presenting him with a challenge.
- contain intrinsic motivating features that intrigue or stimulate the learner to desire the solution.
- offer the opportunity for extension of the problem.
- permit more than one means of solution.
- place the learner in creative, interesting, or stimulating situations.
- ask the learner to make use of more than one skill at a time.
- provide opportunities for verification of the result.
- are not purposefully misleading.

A good problem does not have to satisfy every criterion above. However, most good problems will exhibit a majority of the charactertistics listed.

Generating Good Problems

Writing good problems yourself is one of the most satisfying and useful means of obtaining problems appropriate for a particular level or class of students. To write good problems requires some knowledge of the format and structure of good problems. But the primary requirement is an inquisitive mind, an awareness of your surroundings, and a receptivity to problem exploration. Potential problem situations abound everywhere and only require an open and sensitive eye to to be found.

Suppose you are visiting a local pizza parlor and chance to look upon the sign listing prices. Have you ever considered how a sixth grade student might look at prices compared to sizes? Why are the 16-inch pizzas more than double the prices of the 8-inch pizzas? What does 8-inch and 16-inch refer to anyway? Out of these considerations, an interesting problem can grow.

Grades 6-7 *At Patty's Pizza Parlor*

Type/Size	*8"*	*10"*	*13"*	*16"*
Cheese	*1.80*	*2.75*	*4.15*	*5.60*
Pepperoni	*2.30*	*3.40*	*5.20*	*6.85*
Deluxe	*2.90*	*4.10*	*6.05*	*8.00*

Our family of four likes pepperoni pizza, so we usually order four 10-inch pepperoni pizzas. Josh – he is our biggest eater – said "I wonder if we could get more pizza at about the same price by ordering other combinations?" Susan – she is a little finicky – piped up, "Well, if you are going to do that, how can we get the most crust? I like crust the best."

What combinations could you order to satisfy Josh's wish? Susan's wish? Both?

A problem such as this can make an excellent problem solving project. Deciding what needs to be determined, assigning responsibilities, examining data, and interpreting results are student actions that contribute to group problem solving growth. Extending area and perimeter concepts to a real problem and application of computational skills are added bonuses.

Another rich source of information from which problems can be developed is the newspaper. Since articles generally focus on items that are of interest to readers, an intriguing problem background can often be found involving some nearby attraction. Consider the interest level that your students might have in this rollercoaster problem project developed from the accompanying newspaper article.

Park Unchains Baddest Beast

KINGS MILLS, Ohio — The Beast is on the prowl.

The Beast is the new roller coaster at Kings Island Amusement Park. It was officially and literally unchained Friday.

Starting this weekend, the public will have the opportunity to ride the wooden monster that is being billed as "the biggest, baddest, longest, faster coaster in the world."

During Friday morning ceremonies, three members of the U.S. Marine Corps snapped a link chain restraining the coaster's 40-seat car.

Officials of Taft Broadcasting Co., Kings Island's parent company, then took part in the maiden run of the $3.8 million roller coaster.

The experience lasted 3 minutes, 40 seconds.

The Beast is an appropriate name.

Covering 35 acres, The Beast has 7,400 feet of track, three vertical drops, three underground tunnels and a 628-foot helix — a large circle of track.

The first vertical drop has riders plummeting 135 feet at a 45-degree angle.

By the time The Beast zooms into a tunnel, it is traveling about 70 miles an hour.

Rain dampened Friday's dedication. Moments after the initial run, light rainfall turned to a downpour with thunder and lightening adding effects — and curtailing further runs of the coaster.

Kings Island wanted three things out of The Beast — thrills, speed and smoothness. The amusement center was also concerned about safety.

The Beast has a computerized safety system and closed circuit television, which monitors the entire track. There is also a height requirement — 54 inches.

Persons under 4 feet 6 will not be permitted on the ride.

Kings Island opens for the season Saturday. It is open at 10 a.m. Saturdays and Sundays until May 20. Daily operation will be from May 26 to Sept. 3.

Reprinted with permission from *The Columbus, Ohio Dispatch*

Grades 5-6 Rollercoaster Ups and Downs

a. If coasters leave approximately every 30 seconds, how many trains are on the track at any given time?

b. Suppose each coaster carries 32 passengers. Approximately how many people ride in one day?

c. How far does one coaster travel in one day?

d. About how many people ride the rollercoaster during its regular season of daily operation?

e. About how far apart are individual coasters on the track?

f. About how many miles are traveled by coasters during a regular season?

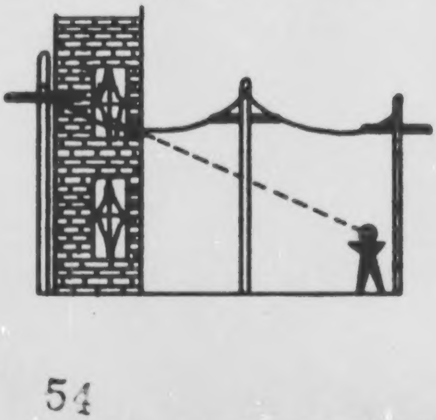
Assessing student interests and activities can also be useful in identifying situations they will find engaging. Asking students to create problems that they find interesting can sometimes generate problems that are worthy of detailed study. Such student creations may stimulate some related problem invention from you. However, you should not restrict students to problems strictly from their environment. Students are just as frequently struck with situations that appeal to their fancies.

Perhaps one of the most fertile sources of interesting and useful problems is that which occurs spontaneously through a teaching methodology that provides rich and interesting learning experiences. For example, sending students outdoors to measure and record items of their choosing with a meter-length string is an open-ended activity that might produce the question, "Mrs. Ortega, how can we find how big around this skinny, little pole is with our meter strings?"

Spontaneous Problem Solving

An example of unplanned problem solving developed during an outdoor learning activity which required the use of indirect measurement to determine the height of a brick building. Students were later asked to count how many bricks high the building was, find an average width of ten bricks (with mortar), and check their indirectly calculated height against the height found by counting and measuring bricks.

The unanticipated problem occurred when students attempted to count the number of bricks in the building's height. Eye fatigue caused blinking to occur which lost contact with the particular brick being counted. A practical problem, calling for imagination, ensued. It was solved, creatively, by sighting across an intermediate object, an electrical wire, and walking backward slowly. The line of vision was retained through blinking by the wire.



Problems which occur spontaneously model applied problem solving. Students gain experience in handling problems that come up in unforeseen ways. When a practical situation develops a snag, they witness the ingenuity and, perhaps, frustration that accompany realistic problems. Or in tackling a problem which grows out of a classroom discussion, they become alert and receptive to problematic situations and ideas that deserve further consideration and analysis.

On finding problems, George Polya notes, "Good problems and mushrooms of certain kinds have something in common, they grow in clusters. Having found one, you should look around; there is a good chance that there are some more quite near."

Accordingly, you should examine good problem situations to determine whether they are indeed exhausted and also whether another good problem may be found through variation of the original conditions. Polya suggests that variation by generalization, specialization, analogy, decomposing, and recombining are sometimes fruitful methods of generating new problems.

Problem Resources

Problems can also be found through materials and other people. This publication contains more than 200 problems suitable for classroom use. The appendix of these materials lists books on problem solving, puzzles, teaching, and other areas that are rich in problems appropriate for different grade levels and circumstances. Some smaller publishing houses seem to specialize in supplementary materials which abound in good problem solving materials.

Professional meetings and inservice experiences are other good sources of problems. Nearly any regional, state, or national mathematics meeting will have special sessions devoted to problem solving. Most speakers use excellent problems in their presentations. Local inservice meetings that are oriented toward mathematics provide similar opportunities for locating good problems.

Some state and federally funded projects are rich sources of problem solving materials. Two such projects in Oregon are the Mathematics Resource Project (materials now available commercially) and the Lane County Problem Solving Project. Another large problem solving project has been the joint enterprise of several universities in the state of Iowa. A little investigation through local officials or through a national resource such as ERIC* can identify projects that may be useful to you.

Doing a little research at a nearby university can also be a fruitful activity. Research reports, PhD theses, psychology materials, old textbooks, and university educators can be sources of good materials. Most persons interested in problem solving have pet problems that they are willing to share and interchange with other interested parties. Communicating with colleagues at the district or other levels can begin a flow of information exchange that may continue to pay dividends over the years.

Many good problems are to be found in materials which may not carry a mathematics label. Some of the most creative, challenging, and worthwhile problem situations arise out of interdisciplinary materials and activities. Aspects of these situations can be used in mathematics. Or a whole situation may be studied meaningfully to show how mathematics plays an integral part in such activities.

Not to be overlooked are the students, themselves. Besides creating problem situations as part of an assignment, students have access to problems from home, acquaintances, work, and play that are frequently worthwhile. When students realize that problem solving is a very important aspect of their school work, they will begin to bring such materials to school and to the attention of classmates and teachers.

*There are 16 ERIC Centers (Educational Resources Information Clearinghouse) nationally. The ERIC Center listing mathematics education as an area of specialization is located at The Ohio State University, 1200 Chambers Road, Columbus, Ohio 43212.

PRIMARY INSTRUCTION *

In many textbook series, problem solving has been ignored in grades K-2 and, sometimes, in Grade 3. In most series and curricula, no identifiable problem solving strand exists in which children are made aware of strategies or alternative ways of solving problems. Although the number of creative and open-ended problems conducive to problem solving instruction seems to be increasing, there are not enough of these problems present in textbooks to conduct a program with a sound problem solving strand.

Finding answers to word problems in textbooks may or may not involve problem solving. If the problems are so easy that children know the answers automatically, there is really no problem at all. Many word problem lessons follow a computational topic and involve exclusive use of this particular operation. In such a setting, children may be learning to transform such a problem to an open sentence or a number sentence. But repeated use of such lessons will not usually be productive of problem solving thinking and may be indicative that such lessons are designed more for computational practice and refinement than problem solving.

Many of the best problem solving situations in the primary grades come from everyday situations. "How many more chairs will we need if we are having five visitors and two children are absent?" "How many milks will we have if the boys order two chocolates and the girls order one vanilla and one chocolate?"

Generally, problems should be used throughout teaching units from instruction through reinforcement. Problems should be "real" as often as possible and be interesting enough that the child wants to solve the problem. Fun problems such as, "How many hamburgers would it take to stretch from our classroom to the cafeteria?" are useful in modeling number ideas and operations. Children particularly need the experience of relating a real-life and practical occurrence to a mathematical representation of that occurrence. They also need the experience of visualizing real situations to fit numerical situations.

Care needs to be taken not to underestimate the abilities of primary children to solve problems, often using techniques that we are unable to predict. Children in the primary grades are increasingly able to make estimates, keep records of their observations, make symbolic statements to fit events or situations, and make decisions on the basis of what has or may happen in a given situation. The richness of the problem solving environment and the quality of problem situations to be solved are proportional to the kinds of learning experiences that children derive from solving problems.

Children must also believe that they can solve a problem. Moreover, they must be confident when they have a solution. This kind of confidence must be developed in an atmosphere created by the teacher in which they feel both free and secure. Problems must be difficult enough to present some blockage, but easy enough to instill confidence. Individually, a child's tolerance time for dealing with a problem before giving up should be gradually expanded. But children should recognize that some problems are not within their capability and not be threatened by ending their problem solving efforts. However, they should feel accomplishment in having tried some things within their capability, knowing that the problem did not yield to those approaches.

In creating a problem solving atmosphere conducive to individual problem solving growth:

- Let children use their own language to express the problem.
- Maintain a learning environment rich in manipulatives and ancillary supplies for ready use in problem solving.
- Let children work together. Discussion of problems, sharing of ideas, debating of alternatives, verification of solutions are part of group problem solving as well as peer learning.
- Develop problems appropriate for different ability levels. Good problems frequently permit different levels of sophistication in their solution.

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*Adapted from Problem Solving/Calculator Resource Packet, unpublished NSF materials, Suydam, et al.

- Provide problems that have no answer or that have more than one answer. Too often, children get the impression that all situations have one and only one exact answer.
- Give children problems for which they must collect or generate some of the information or data. Textbook problems provide all the necessary facts and only the essential information. Everyday experiences in problem solving require selection of relevant information or seeking out data that must be used.
- Create some situations in which a problem is not well-defined or in which the child must define the problem he will try to answer. We often need to ask questions in an everyday problem solving situation before we understand the nature of the problem that we must solve.
- Use skillful questioning to help children sense what a problem is about, to form a mental picture of the relationships or patterns, and to systematically determine procedures and alternatives.
- Discuss, without solving, problems, reasonable answers and estimates, alternative procedures for finding solutions, variations of problem situations.
- Select problems which provide for maximum pupil involvement and minimum teacher guidance.
- Break complicated problems into manageable parts when frustration approaches. But be alert to exercise sufficient restraint that the challenge (and resultant satisfaction) is not also removed.
- Compliment alternative solutions or approaches to problem solving situations. Let children know that you encourage different ways of thinking about problems.
- Select problems that reinforce and use computation skills and numerical concepts without those skills or concepts being the primary focus.
- Be flexible in permitting children to continue with inaccurate or inefficient results or procedures for a time. Avoid being overly didactic or the source of all answers or answer verification. Encourage children to look to their own resources for answer validity.

Problem solving need not be considered dependent upon reading ability. Children should be given some problems orally before and after they have learned to read. Real-life problems come in many forms, but seldom packaged neatly in words. A problem situation which is orally described rather than concisely stated in written form may assist students to focus more upon the aspects of the problem rather than attempting to identify word cues or a problem form as a clue to the solution.

Children in the primary grades need to begin to develop a variety of strategies for solving problems, and these strategies need to be identified and discussed with children as helpful actions to use in solving problems. The following section lists some problem solving strategies that are appropriate for primary grades as well as illustrative problems that can be solved using that strategy. Strategies will not necessarily match those discussed earlier in these materials for grades 3-9.

Strategies and Illustrative Problems for the Primary Grades

Using drawings and diagrams

You get on an elevator at the main floor.
Go up 7 floors.
Go down 4 floors.
Go up 8 floors.
Go down 3 floors.
Go down 6 floors.
Go up 2 floors.
Get off the elevator. What floor are you on?

Two of each 5 books in the classroom were found to be in poor condition. If 25 books were being used, how many were in poor condition?

Dramatizing or acting out problems

Four children were in reading circle together. Five more children joined them. How many children were in reading circle then?

Nine children were washing the chalkboard. Three finished and sat down. How many children were still washing?

Collecting information

How many times does your heart beat in one minute? while sitting? after running?

If Joey brought two boxes of crackers and cheese dip for the class to have a snack, how many crackers could each person have?

Making up problems

Make up a story using 9, 5, 14, ducks, foxes, corn and a boy or girl.

Give students a table or other set of data (such as a simple recipe). Have them make up problems using the data.

Problems without numbers

Tell how the problem would be solved:

Jill had some money.
She gave some to Roy.
How much money does she have now?

Sue knows how many meters of ribbon she needs and the cost per meter. How can you find how much the ribbon will cost?

Matching mathematical sentence with drawing

Write number sentences on the board.
Draw a picture to match one sentence.
Have students select the correct sentence.

Place several pictures on the board. Have students find the picture that goes with this problem that is read to them:

"I had three cups. I broke one. How many cups are left?"
Find the picture and then write the sentence.

Make up an easier problem

Carl spent $4.37.
Joan spent $5.13.
How much more did Joan spend than Carl?

Carl spent 4¢.
Joan spent 5¢.
How much more did Joan spend than Carl?

Make up a harder problem that could be solved the same way as this easy problem:

Jeanette had 6 balls and some boxes. She put 2 balls in each box. How many boxes did she fill?

Using materials

Yummy Bakery puts six buns in a box. How many boxes do they need for 42 buns?

Have students use buttons to represent buns and tongue depressors for boxes to answer the questions.

For an activity such as art requiring half a sheet of construction paper per student, have a pair of students decide how many sheets are needed for the class.

Determining missing information

If you have enough information to answer the question, find the answer. If you need more information to answer the question, write NMI.

Chickens in South America lay 39 eggs each month.
Chickens in Australia lay 47 eggs each month.
There are more chickens in South America than Australia.

1. In what place do chickens lay the most eggs each year?
2. Where do the people eat the most eggs per person?
3. Where is the total number of eggs laid each month the greatest?
4. How many more eggs per year does an Australian chicken lay than a North American chicken?
5. How many less eggs per year does a South American chicken lay than an Australian chicken?

The sum of two numbers is 15. One number is four more than the other. Find the two numbers.

Is the answer reasonable?

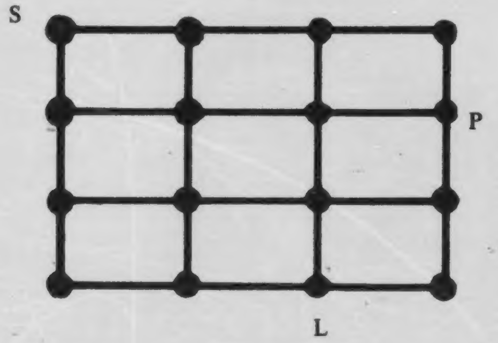
When Florence was 8 years old, her father was 4 times as old as she. Now, Florence is 16 years old, so her father must be 64 years of age.

Phillip weighs 62 pounds standing on one foot, so he must weigh 124 pounds standing on two feet.

Using maps, tables, graphs

Have pupils fill in a table for each child in the class, noting names and distances traveled to school from home. See how many problems or questions the class can make up using the facts given in the table.

On the map below, S is Sandra's house, P is the park, and L is the library. Going the shortest distance,



a. how many ways can Sandra go from home to the park?
b. how many ways can Sandra go from the park to the library?
c. how many ways can Sandra return home from the library?

INTERMEDIATE AND JUNIOR HIGH INSTRUCTION

The scope and sequence development of the problem solving strand of most textbook series classifies problems according to content. Accordingly, most sets of word problems are designed to practice some specific computational process or "problem type." This kind of classificatory problem solving approach can be beneficial in the introduction of applications of these specific concepts and processes to word situations. However, accompanying problem sets are usually limited to artificial situations in which problems are restricted precisely to the information required for the problem. The problem is well-defined, and reading is at a minimum. Problem settings frequently are not relevant or applicable to situations in which students would perceive using mathematics outside of the classroom.

Intermediate and junior high school students need additional kinds of problem solving experiences which provide some other dimensions to their problem solving development. By this age, they are ready to give more specific thought to the kinds of formal processes and techniques which they are using in problem solving situations. Some classroom experiences can be profitably directed toward an awareness of the various stages in solving a problem. We again caution, however, against any kind of insistence that students employ a 4-step or other model to their problem solving efforts. We encourage that attention be given to the importance of certain activities in the problem solving process: use of strategies, awareness of reasoning procedures, attention to problem organization, recognition of attitudinal elements of problem solving, management of personal resources.

Students need more opportunities with problem situations that arise out of their own experience or which have open-ended or investigation aspects. Formulating and developing their own problems is another activity which should be encouraged. Problems arising from the students' backgrounds, particularly ones which require the collection and use of data, are valuable for transfer of problem solving skills from the classroom. Other problem situations may be selected which require the application of specific content to practical situations calling upon interdisciplinary skills. USMES (United Science and Mathematics for Elementary Schools) materials contain numerous examples of this kind of problem situation.

In general, efforts should be made to engage students with problems which are more sophisticated and lengthy. This might entail a grouping of related problems embedded in one problem setting, or single problem which has several easily identifiable subgoals that are necessary to achieve a more important overall goal. A guided discovery lesson can be designed according to this second format. Either approach can be useful in planning for ability differences within a group.

Students should begin to assume more responsibility for their problem solving efforts, expanding their tolerance for frustration and looking to each other and themselves for interpretation and evaluation of results. They need to begin to engage in active hypothesis development, conjecture testing, generalizing, inductive and deductive reasoning, and independent learning. It is desirable for students to change their perceptions of the teacher from that of dispenser of knowledge and answer evaluator to one of a source of interesting ideas, challenger of thinking, and experienced resource person.

Though students are engaging in more adult-like problem activities, they must also be recognized as being in a developmental stage and still requiring many considerations that we might associate with a younger learner. Many need assistance in knowing how to organize their work for a problem, how to construct a table, how to collect and record data. Some will still be at the level of learning to separate their work so that it does not all run together, preventing them or others from seeing relationships among the data.

A variety of materials ranging from odds and ends for manipulatives to sophisticated devices such as calculators, trundle wheels, and balances should be available. Sometimes it is mistakenly assumed that less reliance should be placed upon concrete objects as students grow older, that mathematics should become more of a mental and paper and pencil experience. Too often there is a tendency to rush this process. Students continue to need to use manipulatives, but those which they employ should gradually increase in sophistication. Consumables such as graph paper, string, and construction materials should be readily available at all times, and students should be encouraged to make frequent use of them.

Small and large group problem solving should receive more emphasis in these grades. Attitudes important to problem solving need nurturing, such as flexibility in choosing productive problem attacks. Reinforcement for alternative, creative, or unusual methods of solution should occur frequently. Occasional problem situations emphasizing competitive aspects of group problem solving can be used. The quality of solution rather than the speed of solution might be appropriate evaluative criteria.

Problem solving in these grades should be more than an activity which is restricted to certain periods each week. It should be a teaching philosophy at all times. Problem situations can motivate and precede the development of algorithmic processes, for example, while supplying students with the concrete examples which will enrich the meaning behind the algorithms. The perceptive teacher will be alert to seizing student suggestions for problem investigations whenever they occur. Since such suggestions are likely to involve extensions of content material being studied, it will give students a sense of involvement and satisfaction that they are helping to shape the mathematical experiences they are enjoying.

One of the primary objectives for problem solving during the intermediate and junior high grades should be to provide a variety of problem solving situations for study. Some of the kinds of situations which should receive emphasis include:

- Problems with superfluous or insufficient information.
- Problems that arise from the student's environment or interests.
- Problems which do not have well-defined answers or which involve inexact skills such as estimation.
- Problems posed in nonverbal settings utilizing manipulatives within the classroom or activities outside the classroom.
- Problems developed around central themes which are extended to substantive results but are subdivided into manageable parts.
- Problems requiring students to make choices about the degree of accuracy required or to be used within the problem.
- Multistep problems in which identification of intermediate step meanings are emphasized.
- Problems involving logic, reasoning, testing of conjectures, reasonability of information.
- Problems which require decision-making to result from the mathematical outcomes.
- Problems which emphasize the practical application of mathematics to business or consumer situations.
- Problems requiring the collection and interpretation of data.
- Open-ended and investigatory problems.
- Problems involving modeling to explore interdisciplinary topics.
- Problems which require students to conceptualize and make meaningful very large or very small numbers.
- Problems applying mathematics to school, home, or personal situations.
- Problem situations calling for more than one method of solution.

In selecting problems for student study, three criteria should be used: (1) the problem involves content that is appropriate for study, review, or reinforcement; (2) identifiable problem solving skills or procedures are required for problem solution; (3) the student is required to do some original thinking in the solution of the problem. Using these three criteria and providing a good mixture of problems ranging from real-life applications to purely student interest situations will provide a rich and varied curriculum in problem solving. Following are examples of the kinds of problems listed above.

Insufficient Information

Grades 6-7 Tricentennial Celebration

For the celebration of the 300th anniversary of our nation, Oscar Gomez wants to form a line of people holding hands that stretches from New York to Los Angeles. The mayor of New York will hand a tricentennial message to the first person of the line who will hand it to the second person and so on until the last person hands the message to the mayor of Los Angeles. How long will it take to hand deliver the message?

Student Interest

Grades 6-7 Baseball Standings

Following is the won-lost record of the baseball teams in the National League on July 14, 1965. Arrange the teams in correct descending order of "percentages" without calculating percentages.

	W	L		W	L
Chicago	41	46	New York	29	56
Cincinnati	49	36	Philadelphia	45	39
Houston	39	45	St. Louis	41	45
Los Angeles	51	38	Pittsburgh	44	43
Milwaukee	42	40	San Francisco	45	38

Estimation

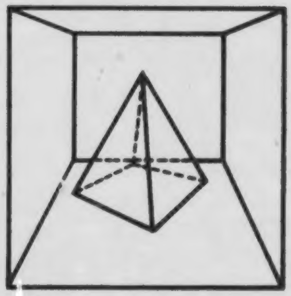
Grades 4-5 Tree Leaves

What means would you use to determine a good estimate for how many leaves there are on a tree?

Nonverbal

Grades 8-9 Pyramid Height

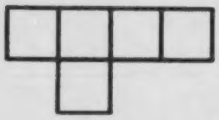
The problem task is to be able to find the height of a pyramid from information which can be determined from the outside. Constructed within the classroom is a pyramid framework which you can get within. Take measurements, determine how the height can be found indirectly, and check your procedure by measuring.



Central Themes

Grades 6-7 Pentomino Puzzlers

(a) Find all the shapes possible that can be formed by joining five squares along an edge (pentominos). There are twelve.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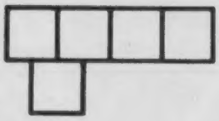
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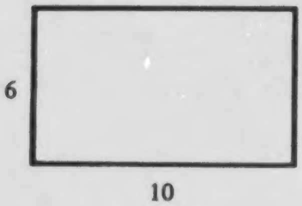
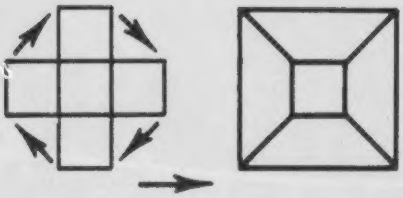
(b) Arrange the twelve pentomino shapes into a 6 x 10 frame; a 5 x 12 frame; a 4 x 15 frame.



(c) Some of these shapes will make boxes without tops. Others will not. Find the ones which make boxes.

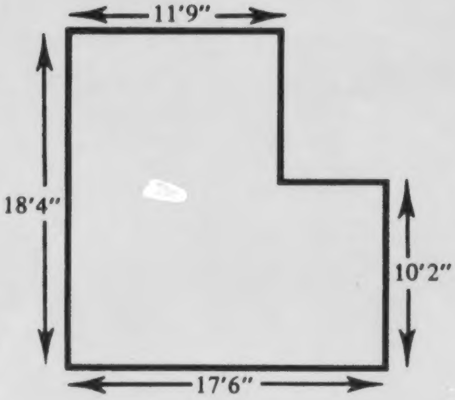
(d) An employee at the box factory ordered sheets that are 4 x 5 from which to make topless boxes. How can you cut one of these sheets to get four boxes? Can you find more than one way?



Decision-Making

Grades 8-9 Carpeting Planning

You have decided to carpet the living-dining room area of your home sketched at the right. You have not yet decided whether to use square foot sections or order from a roll which is 12 feet wide. Determine the amount of each that you would need to order should you make that choice.



Multistep/Intermediate Result Identification

Grades 6-7 What was that Speed?

The average time for a car to pass between the endpoints of a 100 m strip alongside our school's street was 67.9 seconds. Determine this speed in km/h and label any intermediate results with units (e.g. km/min).

Reasoning Problems

Grades 5-6 Balancer's Delight

One of nine objects, identical in appearance, is slightly heavier than the rest. With only an equal arm balance available, how can the heavier object be determined in just two weighings?

Practical Applications

Grades 8-9 Concession Stand

Your club will again handle concessions at the home football games this year. You need to place an order for food and drinks for the first game sales. Last season's total attendance for six home games was 13,593. Total food and drinks consumed were: 9,854 hot dogs, 6,512 hamburgers, and 21,678 soft drinks. This year's team is better so attendance is expected to be at least 15% higher. But price hikes for food are expected to lower demand by about 5%. How many hot dogs, hamburgers, and soft drinks should you order for the first game?

Multiple Solution Methods

Grades 4-5 Flagpole Height

Describe at least three different methods for finding the height of the school flagpole. Assume that it is a cloudy day, that you are not allowed to climb the pole, and that there is no ladder available that is long enough to reach to the top.

Accuracy Decisions

Grades 7-8 A Football's Volume

Describe and carry out a method to find the volume of a football (other than by water displacement). Check the accuracy of your method by determining the ball's volume a second time by using water displacement techniques.

Data Collection and Interpretation

Grades 5-6 Favorite Movies

Conduct a poll of students in your school by classes to determine their favorite movie seen this year. Organize your information into some kind of graph which best conveys the point you wish to make. Make a list of conclusions which you feel are justified on the basis of your information. Make another list of related items that you feel that your information **does not answer.**

Investigatory

Grades 5-6 Coin Flips

On the average, how often will two flipped coins come up heads? three flipped coins come up heads? four flipped coins come up heads? Can you **predict** how frequently five flipped coins will come up heads? How do the results for three coins coming up heads compare to a single coin being flipped and landing heads three times in succession?

Modeling

Grades 7-8 Population Studies

In social studies, we studied the effects of two-children families versus three-children families in population education. Your task is to create a model of two-child family growth versus three-child family growth. Assume that two couples, age 20, without children, are placed upon the island of Fertilia. These couples and all of their descendants are committed to having only two children per family. At the same time, two other couples, also age 20 and without children, are placed upon the neighboring island of Greater Fertilia. They and their descendants agree to have exactly three children per family. Show how the population will grow on these two islands over a period of 160 years. Make whatever assumptions you wish about when children are born, when people die, and so forth. But remember that people die as well as being born and, therefore, have to be accounted for.

Applications to School

Grades 5-6 Organizing a Campaign

To protest the cafeteria food, Grumpy Smythe decided to send a note to the principal and the next day to encourage three other students to do the same. Each of these students is to find three different students the third day to write notes and so on. How many protest notes will the principal receive on the 10th day? How many notes in all will he have received by the end of the 10th day?

Conceptualizing Large Numbers

Grades 6-7 Clean Up Your Act

A sixth-grade classroom has decided to clean up the environment while learning how large one million is by trying to collect 1,000,000 twelve-ounce pop cans in their school. If they are successful, determine how much space 1,000,000 cans will require.

CREATING A PROBLEM SOLVING ENVIRONMENT

The learning environment of the child can limit or advance the instructional potential of a learning activity. If problem solving is an important program goal, then the setting and organization of the learning area will reflect this importance and convey that message to the child. To support a particular learning activity, the environment should be attractive, motivational, purposeful, functional, and provide for flexibility.

In a learning area where problem solving is a significant activity, one might expect to see quotations about problem solving and goals for student achievement. Interesting and challenging problems are prominently displayed. Student problem solving work is highlighted in a conspicuous place. There is visible evidence of student progress in the form of charts or recognition devices. Biographical materials on prominent problem solvers and their achievements may be exhibited. The learning area is organized in ways that permit different problem solving activities to occur. Manipulative materials, supplementary information, and other ancillary devices about problem solving and conducive to that activity are present.

Problem Solving Centers

Solving problems is perceived as a challenge by most students. This allows teachers the opportunity of planning small group and individual student learning activities that capitalize upon this interest. A learning center attractively and appropriately planned can be a place where students can spend some free time or be assigned by the teacher to do interesting and beneficial problem solving work. The center can house the large assortment of manipulatives and other devices that problem solvers may wish to employ. It can be the site where a filmstrip projector is set up with problem solving materials. Books or other printed material from the media center or a classroom collection can be located there. Challenge problems or situations can be established and changed regularly as an optional activity for students who "accept the challenge."

Perhaps the most important function of the problem solving center is to provide an area where a student can work on a problem with another student, be stimulated by the peer exchange of ideas, and experience group problem solving. It is a place to be motivated, to have your interest stimulated, and to engage in a stimulating activity in a nonthreatening environment. A creative name like "Sleuth's Corner" will make it one of the most popular spots in the learning area.

Bulletin Boards

An effective bulletin board can be an efficient way of increasing enthusiasm for problem solving work as well as utilizing nondirected instruction. Earlier, we cited problem solving activities as requiring more time for instruction than lecture-recitation techniques, and that an adjustment might be necessary within the curriculum to devote more time to problem solving. A bulletin board can be used for some activities that might otherwise take class time.

Recognition of work is an important aspect that can be handled well through a bulletin board. Student work can be featured from time to time that emphasizes different aspects of problem solving: creative solutions, specific strategies, neat, organized solutions, multiple solutions, and so on. Through skillful selections, a teacher should be able to highlight every student's work at various times.

A bulletin board can also be used to present challenge problems. The **Problem of the Week** or **Day** can be a regular feature of a selected bulletin board in the learning area. Students may be permitted to put up their solutions to the **Problem of the Week** with challenges coming from classmates for incorrect solutions. Additional solutions may be added if they employ different techniques. Using a bulletin board in this way, students gain experience with self-evaluation, multiple problem solving approaches, and outstanding solutions without taking from class time.

Problems as Rewards/Challenges

Within the proper learning environment, a teacher can direct problems to individuals as rewards for their accomplishments or as challenges to their abilities. Frequently, we tend to consider learning activities in relationship to whole classes or subgroups within the class. However, problem solving abilities within a class are likely to vary substantially.

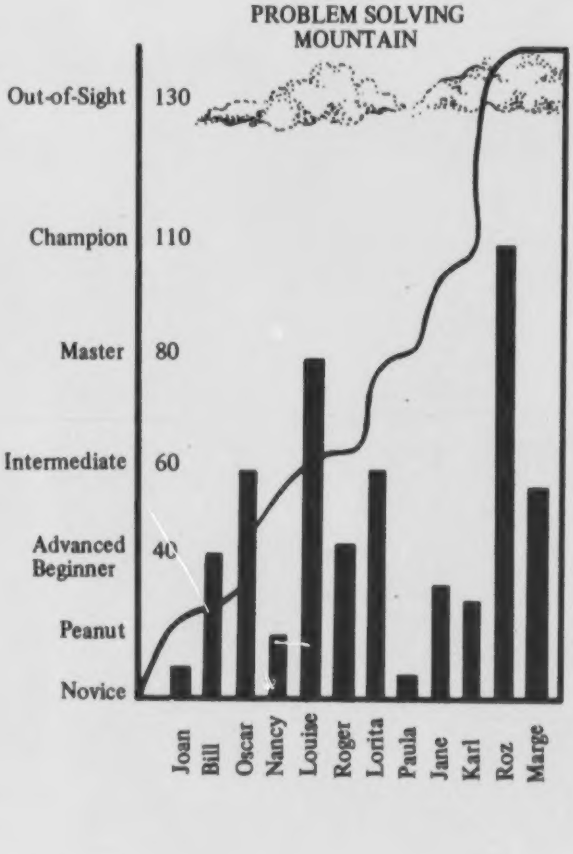
By giving particular problems to individuals, several objectives can be accomplished. The problem can be tailored to the abilities of the individual so that it offers him a challenge within his reach. Since no other student has that particular problem, he is not pressured to compete against another person in a time or accuracy sense. Motivation runs high because it is his problem, no one else's. If the problem is to be solved, he must do it. The teacher selected this problem just for him; therefore, the teacher is expressing confidence in him as a problem solver in being able to solve that problem.

Since the selected problem is uniquely for his consideration, he can experience the satisfaction of sharing it with others (after he has worked with it, had time to analyze it, and be the 'expert'). Of course, the sensitive teacher will recognize that every student should receive such rewards or challenges equally. Less capable students will particularly benefit from the affective reinforcement associated with this teaching technique.

Gimmicks for Accomplishment

Older students as well as younger students enjoy a certain amount of hoopla to accompany learning experiences, especially enjoyable ones. They also like the spirit of friendly competition and the recognition that accompanies it. A creative teacher can accomplish both objectives through a number of devices. Placing point values on problems in a problem file, letting students elect those problems which they wish to tackle, and keeping a record of their progress on a centrally located chart can encourage students to work on problems on their own time. Certificates *(Problem Solving Excellence)*, titles *(Problem Solver Extraordinaire)*, traveling tokens *(Sleuth of the Week)* can be utilized effectively. Even with older students, 'campy' awards *(a piece of candy)* will be received humorously, but nevertheless sought.

If competition is likely to cause embarrassment or anxiety, aliases of famous mathematicians can be employed. Progress will be noted for Gauss, Euclid, or Fermat. Only John, Susan, and Laurie will know the identities behind those pseudonyms. The mystery and secret competition as progress is updated will provide an excellent stimulus for students to learn more about these famous persons.



Problem Files

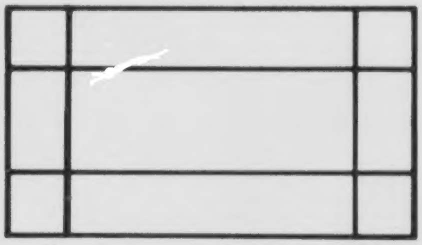
At the heart of any good problem solving strand is a series of excellent problems. Mediocre or poor problems simply will not do. Nor will problems of only one type or one degree of difficulty. To operate an effective problem solving component, a teacher needs a large number of problems that are keyed to different content, involve various strategies, represent varying degrees of difficulty, and relate to various areas of student interest. Although problem sources abound, few are classified according to grade or ability levels. Such levels are also likely to vary from class to class. Experience suggests that personal selection of problems generates the most appropriate files for teaching use. Such lists continue to grow each year as problems are shared among teachers, modified by student feedback, added to from new resources, and written during creative moments.

The key is to have a good filing system that makes problems readily accessible for teaching or for student use if they are permitted to work with a file directly. Once problems are added to one's personal file, he will find that they are old friends that he delights in sharing with new students each year and which continue to amaze him in the way that creative, eager young minds are able to respond in unique and unusual ways.

Problem Solving File Card

Grades 6-7 Difficulty 5 pts

Rectangle Madness



How many rectangles can you find in the figure?

Strategies: Organized List
Simpler Case

Content: Geometry

ORGANIZING THE CURRICULUM

Problem solving though deemed important by teachers has not received its due because it lacks clear definition in the curriculum.

The conjecture advanced above reflects the current state-of-the-art for problem solving. Most teachers support the notion of the importance of problem solving but have not initiated strong problem solving programs in their classrooms. Textbook authors and publishers of other educational materials have been reluctant to develop explicit problem solving strands within their materials.

The problems in developing a curriculum for problem solving are many. Students develop problem solving capabilities at different rates. They approach individual problems in different ways. Success with solving one type of problem is no assurance of success with another. Problem solving involves affective as well as cognitive dimensions. It takes more time than lecture-recitation instruction.

Research suggests that an explicit understanding of how a person develops problem solving capabilities is not well understood and not necessarily the same for all individuals. Somehow it involves exposure to good problems, awareness of various techniques, knowledge of approaches to thinking, intervention by good teachers, and interaction with other students.

These are problems associated with teaching and evaluating problem solving. However, they should not become excuses for postponing problem solving instruction for more easily attainable objectives, for fuzzy teaching, or for lack of curriculum planning. It may not be possible to specify a precise recipe for developing good student problem solving, but we can assure that instructional activities and time are planned and coordinated in such a way that individuals have the opportunity to realize their problem solving potential, whatever the mechanism.

Problem Solving Skill Instruction. Students should be exposed to a variety of problem solving techniques, strategies, and modes of thinking as part of a planned and sequenced set of activities. At different grade levels, particular problem solving skills should be formally introduced and practiced. A strategy such as *look for a pattern* may be used informally as part of mathematics instruction in grades prior to the targeted grade. However, the key to a total problem solving curriculum is to give a skill particular emphasis and identification **as a problem solving tool** at a particular grade level. Thereafter, it will be practiced and used in subsequent grades as a problem solving skill rather than a good thing to try, selected by happenstance.

Planning a scoped and sequence listing of problem solving processes and skills will assure that students are exposed to a range of such procedures, have an opportunity to practice them as problem solving tools, and avoid the likelihood of hit-or-miss, incidental development of such important learning outcomes.

Related/Subordinate Skill Identification. Many problem solving skills and processes are dependent upon simpler empirical skills. In order to use *make a table* as a strategy to help solve a problem, students need previous practice with such related skills as reading a table, interpreting information from a table, constructing a table for given data, and pattern recognition before *making a table* becomes an effective means to unravel a more difficult task. For strategies and other problem solving skills to become operational tools in problem solving, students should have previous opportunities to acquire related subskills prior to the use of that strategy in problem solving.

However, it is not necessary to have student mastery of these skills before their use. Problem solving situations may provide the motivation and means to refine these skills even as they are being applied. Consideration, however, should be given to where these skills are introduced in the scope and sequence of the total program.

Types of Problems. The kinds of problems that are introduced to students are just as important to problem solving development as techniques for solution. Students need exposure to a wide variety of problem situations to enhance their capabilities of responding to the many and varied situations they will face throughout their careers. Textbooks tend to classify problems according to content or subject matters e.g. time-rate-distance problems or percent problems. We are suggesting an added dimension that concerns problem format and structure: single-step, superfluous information, inexact answer problems. Coupled with the classification system of texts, a rich and varied program should result.

Traits and Behaviors. Affective dimensions of problem solving are more difficult to plan activities for. In some instances, it is possible to design learning situations to encourage problem solving behaviors such as *conjecturing, predicting*, or *drawing conclusions.* However, other behaviors and problem solving traits may be more difficult to address through planned experiences. *Self-reliance, risk-taking, creative thinking, interacting* are examples of affective-related behaviors and traits that are fostered through a secure classroom environment, reinforcing teaching, and individual encouragement.

From a curriculum and teaching perspective, it is important to recognize the nature of these affective behaviors and to appreciate the role they play in problem solving. Traits such as *critical judging* may appear less threatening or bothersome when perceived as an integral and desirable aspect of a student's active internalizing and evaluating of a problem situation. Such feedback is certainly more desirable than the submissive response of a passive learner. Identifying traits and behaviors supportive of problem solving will further open up opportunities to instill and nurture such capabilities as they appear in learning situations.

Time. One of the most critical considerations to realizing student problem solving growth is provision of the necessary time for it to occur. Problem solving is an integral goal of the K-9 program, not a secondary by-product. As such, it deserves a significant allocation of time. Formal planned problem solving instruction should consume approximately 15 percent of instructional time. In other words, problem solving should be the major focus during at least three instructional periods every four weeks. Other opportunities for informal problem solving discussions and activities will occur through instruction targeted toward other goals. And a problem solving teaching approach can be used for any content. However, it is vitally important that specific time be allocated when problem solving is the central focus of instruction, giving it the recognition that will make it a desirable learning goal for students and a pursuable teaching objective for teachers.

Related Skills–Other Disciplines. Many of the elements described in a problem solving curriculum for mathematics are common to other disciplines. Just as problem solving attempts to integrate knowledge in mathematics from several sources and apply that knowledge to new situations, we should view the similar need of articulating problem solving objectives across disciplines. In the elementary grades, positive reinforcement and transfer can occur through indication of how a mathematical idea is useful or related to a situation in science. Recognizing that "truth in advertising" discussions in social studies utilizes similar reasoning and thinking skills to those studied in mathematics will have a similar positive effect.

In grades where departmentalization or specialization of teaching occurs, efforts to discuss commonalities with teachers of other disciplines can be productive to identify areas where positive references can be made to work ongoing in other classes. Coordination of teaching or even teaming might be a possibility.

Subject-centered instruction should not obviate the possibility of using good problem situations that might draw upon several disciplines. Many excellent problems are not just mathematics, just science, or just social studies. To avoid using them in a mathematics class (or other class) for that reason is short-sighted. The fact that they are interdisciplinary probably makes them potentially more valuable as a learning activity for students for that reason. Problem solving in real life does not entail classification by subject matter but may well draw upon skills and information from a variety of sources.

The coordination, integration, and cross-hybridization of problem solving development among disciplines is enhanced by having a well-defined, specific, sequenced set of problem solving objectives in mathematics. The following scope and sequence chart represents an attempt to incorporate many of the ideas discussed above. It is intended as a model, not to be adopted in its entirety. A particular school district should make changes in it, moving topics around to fit their curriculum, adding and deleting. Moreover, it should be considered as only part of a more comprehensive scope and sequence for mathematics and general education.

Cognitive Considerations of Problem Solving

Grade	Subordinate/Support Skills	Problem Solving Skills	Problem Types
K	Sorting according to specified attribute Matching 1-1	Use manipulatives to answer picture problems	Posed by pictures
1	Collecting data Record data by tallying Data and pattern observation	Sequencing picture problems Interpreting picture problems Making up problems from pictures	Involving classification Requiring matching
2	Write number sentence from story problem Complete a pattern (sequence) Use a ruler for measuring, record data	Act it out State what is to be found Identify given information	Without numbers Involving number patterns and sequences Using manipulatives
3	Read/interpret a table, bar graph, calendar Use a ruler to draw line segments and plane figures Read carefully and follow directions Read and use a balance, record data	Use a model Look for a pattern Restate the problem Choose correct operation(s) Employ a 4-stage problem approach	With more than one solution To write by students Posed as games/fantasy With single-step solutions Requiring reading carefully/following directions
4	Use a replacement set in an open sentence Estimate answer to exercises and check by computing Test solution by checking Construct table, chart, graphs Sorting and organizing data	Guess and check Construct a table Make a drawing, figure, or graph Determine reasonability of result Use replacement set in open sentence Estimation	With no easily identifiable answer Requiring collection and use of data Involving personalized answers Requiring interpreting charts, graphs, maps Involving making a drawing or list
5	Choose and use a formula Rounding a result Locating points on a number line, coordinate system Recording outcomes of events Read and use measuring devices (nonruler)	Account for all possibilities Identify wanted, given, needed information Write an open sentence Check the solution Rounding	With no solution With superfluous information Employing guided discovery With multistep solutions Using diagrams Posed as mini-problems
6	Write flow chart to list alternatives and procedures Read and construct scale drawings Simple constructions using compass, protractor, ruler Use other resources to find information (nontext)	Work backwards Find another solution Select appropriate notation Apply a formula	Requiring the use of formulas With insufficient information Utilizing discovery
7	Select variable to represent unknown quantity Translate a problem into a mathematical sentence	Solve a simpler or similar problem Identify a subgoal Find another way to solve it	Open-ended exploration Involving real world applications Requiring estimation/approximation
8	Determine precision of result Draw conclusions and generalizations Determine uniqueness of solution Solve simple linear equations	Change your point of view Check for hidden assumptions Use a variable to represent an unknown quantity Generalize	Involving faulty logic/false inference Requiring deductive and inductive reasoning
9-10	Check misuse of data Graph linear relations on a coordinate system	Retrace/record the solution process Determine the uniqueness of solution Determine the precision of solution Determine necessary and sufficient conditions Draw conclusions Hypothesize Define the problem Interpret information Check for misuse of data Use counterexamples	

Affective Considerations of Problem Solving

Traits

CURIOUS — eager to investigate, to learn new approaches/techniques, to understand how a problem is solved

KEEN — interested in problems, quick to respond in class discussions and to individual challenges

INTERACTIVE — participates freely with others, seeking and sharing ideas

CREATIVE — responds to problem situations in new or unusual ways; not confined in problem approaches or ways of thinking

RECEPTIVE — willing to listen to and consider ideas of others

APPRECIATIVE — enjoys and values the importance of problem solving activities, good problems, sound, clever, or different solutions

EXCITABLE — pursues a problem with great energy and intense interest that shuts out other activities temporarily

INTUITIVE — able to act on hunches or educated guesses

RETENTIVE — draws upon and applies previously acquired information in new situations

SELF-CONFIDENT — believes that skills and abilities are adequate to meet the challenges of new problems

RELISHES CHALLENGES — desires and enjoys pitting his abilities against problems

CRITICAL — evaluates ideas and explanations carefully; challenges classmates or teacher on incorrect or partial results; notes and looks for exceptions to generalizations

RESILIENT — able to experience unsuccessful results without negative repercussions on his self-confidence

ORGANIZED — approaches problems systematically, investigates problem ideas in an orderly, sequential manner; keeps a record of successful and unsuccessful attempts

TOLERANT — listens to ideas and problem approaches that are not personal choices; willing to bide time in making and seeing suggestions acted upon; respects problem solving efforts and achievements of others

RESOURCEFUL — able to overcome obstacles in more than one way

FLEXIBLE — capable of changing or expanding thinking to incorporate new or different ideas from others

SELF-DIRECTED — motivated from within to pursue and continue with challenges

INTROSPECTIVE — considers his own thinking processes in problem solving; reflects upon how new knowledge or discoveries integrates with previous information or thinking

RISK-TAKER — unafraid to be wrong in his ideas or to be unsuccessful in efforts to solve a problem; willing to forward ideas about a problem to others for evaluation

Behaviors

QUESTIONS — expands upon problem solving discussion by asking about other cases, how the situation varies by changing givens; pursues matters that need clarification in his own or other's thinking; may challenge the need or application of learning content

NOTES DETAILS — considers all information that may affect the outcome of a problem; alert to recognizing relationships among variable quantities

DISCRIMINATES — perceives similarities and differences among objects or relationships which are important to the problem; distinguishes relevant information from irrelevant problem material

RECOGNIZES PATTERNS — detects similarities that characterize a set of information; able to predict missing elements

ANTICIPATES — examines alternatives using cause and effect reasoning without carrying action to conclusion; capable of meeting problems before they arise

PREDICTS — foresees or foretells the outcome of or results to a problem based upon previous background, experience, or reasoning

GENERALIZES — extends the results of a particular problem or set of data to a larger and more general sitatuion

VISUALIZES — forms mental images of problem variables to perceive interrelationships among them

INFERS — examines problem information carefully to derive hypotheses and draw conclusions

SPECULATES — reflects upon and reasons about problem components, interrelationships, and implications; forms educated conjectures from available evidence

CONCENTRATES — summons all of his skills and resources to attack a problem; overcomes extraneous influences and distractions

SYNTHESIZES — integrates individually acquired skills and information into a larger understanding of the processes and components of problem solving

DRAWS CONCLUSIONS — able to bring thinking to a decision to direct problem solving actions; able to summarize the results of problems or implications

DELIBERATES — recognizes the appropriate times to consider carefully the information of a problem before acting, the implications of a result before generalizing, the alternatives before choosing

PERSEVERES — persists with a problem despite lack of success, discouragement or opposition to his ideas, reluctant to "give up" on a problem

MAKES REFINED JUDGMENTS — able to adjust his thinking or statements based upon additional information; able to improve the work of others by noting subtleties, distinctions, exceptions, or special cases.

USES DIVERGENT THINKING — able to perceive more than one implication or consequence to a problem action; able to consider unique or unusual approaches or outcomes to a problem; able to expand his thinking throughout a problem rather than narrowing it

CALCULATORS IN PROBLEM SOLVING

The hand calculator is a recent product of technological change that now rivals automobiles, television sets, and refrigerators in availability to the average citizen [Bell, 1978, p. 405]. Its capability to carry out many kinds of computations tens and hundreds of times faster than by hand affords the potential for new applications that we are just beginning to realize.

One of these applications resides in the area of problem solving. Since many problems involve calculations, the hand calculator provides the means to "speed up" one aspect of the problem solving process. It also extends our capabilities for dealing with certain kinds of problems that we may not have been able to consider before.

> "I just got a calculator and (bad news) I find it is making me sinfully lazy about doing things I used to do in my head or with paper and pencil, but (the good news) I now tackle things with my calculator that I never would have thought of doing before." [Ibid, p. 406]

As the quotation above suggests, use of calculators in education brings with it certain new capabilities but also suggests changes in the way we have proceeded in the past. The calculation strand of the mathematics curriculum has been such a revered goal of traditional school programs that many educators perceive use of the hand calculator as a threat to our time-honored way of doing things. Their perceptions about calculators are colored by all-or nothing arguments on their use by school children. Fears that hand and mental computational abilities will vanish are verbalized by such questions as "What will the student do when the battery goes dead?" However, such questions are no more germane to effective utilization of hand calculators than, "What do you do when your pencil point breaks?"

The key issue is that we use various kinds of computation techniques to fit certain situations. The act of making a computation is not an end in itself but the means for acquiring additional information or for making a decision. In school programs, we create somewhat artificial situations in which skill-building temporarily becomes the end for that particular instructional activity. In such circumstances, if the objective is to develop hand or mental computational abilities, then use of hand calculators may reasonably be restricted. However, at other times, our teaching objective changes to one of using computations to accomplish some other goal. If the time involved to make these computations by hand is lengthy or if the difficulty of the computation is likely to direct attention away from a more important idea, then use of the calculator becomes a viable means of facilitating instruction.

Another aspect of the concern over using calculators, involves the use of story problems from textbooks. Such written problems are, more often than not, used for more than their problem solving potential. A substantial reason for studying story problems is motivated by the opportunity to practice and refine computation skills in a different setting than that of an exercise format. In such instances, calculator use is not appropriate.

There are, however, a number of situations where calculators can play the same kind of role as flashcards, rulers, or other manipulative devices. Their use enhances the learning activity, making it richer and more meaningful than it would have been with a calculator. These are the learning situations that we wish to focus upon.

A Calculator Should Be Used When

- It extends personal capability to do problem solving.
- It speeds up a search for patterns.
- It enables more examples to be considered.
- It removes the anxiety of computation failure.
- It permits attention to be directed to other problem aspects.
- It provides motivation and confidence that a problem can be solved.
- If offers access to mathematical ideas not available without it.
- It eliminates tedious or routine computations.
- It allows problems of greater intricacy to be attempted.
- It contributes to further applications and explorations of related topics.
- It saves time within the curriculum without significantly detracting from other objectives.

The speeding up feature of the calculator permits students the luxury of considering more cases and examples to test conjectures or to expand upon understandings. Using hand computations a student might be able to consider two or three cases before generalizing. Using a calculator, he might consider up to ten situations **and still save time!** Being able to consider more problems in the same time span can improve insight as well as be a timesaver for a curriculum that is already overburdened.

Confidence has been shown to be strengthened in many situations where calculators are employed. Students who know that they possess computation weaknesses are frequently reluctant to attempt problems that are dependent upon some measure of computation prowess. They know that their possibilities for success are too limited to make their effort worthwhile. Use of a calculator can remove some of this anxiety and initially provide them the confidence that they can solve such problems. Of course, incomplete understandings and learning gaps will still be hurdles to their success. However, the speed of a calculator may offer them the means to use techniques more consistent with their learning styles. For example, trial and error, consideration of other cases, and checking all possibilities are all means that become more practical with a calculator.

The capability to perform computation by machine in a short time interval makes it more feasible to direct student attention to noncomputation aspects of a problem For example, if 85% of the time spent in solving a problem resides in the computation leading to an answer, most individuals are usually somewhat reluctant to examine that result too closely for reasonableness. They may discover that most or all of their work was in vain and has to be repeated. Contrast that situation to one in which computational time constitutes only 20% of the total time spent on the problem. The problem solver's receptivity to examining his answer closely and perhaps readjusting it as a result of that examination is much more apt. Shortening of the time interval between initiating one's plan of action to solve a problem and obtaining a result also enhances a more global perception of the problem and the process used in solving it.

Arithmetical calculations seem to lose their appeal after a certain level of complexity is reached in calculations or degree of difficulty of a problem. A student must desire the solution strongly to make him want to solve a complicated and exacting problem. If the amount of work involved in generating an answer outweighs the satisfaction in that result, the student is not likely to appreciate the problem solving activity.

The calculator can eliminate the tedium and drudgery of lengthy computations that stand in the way of more interesting results. It can also give the learner access to more realistic problems. Most real situations involve "dirty data" that does not always produce an integer or simple fraction answer. Because of time, space, and usage pressures, textbook problems have been forced toward artificially "nice" problem situations to the extent that a student often suspects that he has an incorrect result when his answer is not a whole number or simple fraction.

Intriguing problems that computationally would not be within reach of many students become possible when calculators are available. Consider the following problems.

Grades 5-6 *An Invasion Attempt*

The perimeter of the State of Ohio is 1505 km long and the population is 10 652 017. To guard the State from invasion, how many centimeters of perimeter would each person have to guard?

Grades 7-8

Farming Lake Erie

In a reclamation project, suppose that Ohio has decided to fill in Lake Erie and farm that area. Determine the average depth of soil that would have to be removed from the State of Ohio to accomplish this project given that the area of Lake Erie is 25 688 km^2*, the area of Ohio is 106 211* km^2*, and the average depth of Lake Erie is 17.5 m.*

Adapted from **The Mathematics Teacher**, November 1977, p. 654, Litwiller and Duncan.

Problems involving more difficult numbers can be attempted when using a calculator, and problems which contain more steps or which involve more data can be handled more efficiently. Exposure to such problems is not only intrinsically interesting but suggestive to students of other kinds of exploratory activity they can undertake on their own.

Perhaps, one of the more significant applications of the calculator to problem solving involves the opportunity of introducing and exploring ideas and topics that would not be possible otherwise or which would be delayed until much later in the curriculum. With a calculator available to do the computation work, many interesting and worthwhile topics become available.

> ". . . the most heartening thing was to see how the calculators seemed to encourage children to explore topics not usually studied extensively in sixth grade, such as probability, exponents, sequences, prime numbers, palindromes, negative numbers, division by zero, divisibility, and permutations." [Sullivan, 1976, p. 552]

In earlier discussions it has been indicated that problem solving instruction takes a significant amount of time to accomplish and that time must be found in the curriculum to conscientiously pursue problem solving goals. Use of calculators can cause some shrinkage to occur in curricular objectives as well as the amount of time that is required to cover those topics. Therefore, the calculator may well be the kind of change agent required to provide more time to pursue problem solving outcomes.

> "Universal use of a calculator may shrink the curriculum in some ways but it will certainly expand the curriculum in a problem-solving direction." [Gawronski, 1976, p. 512]

Because calculators are a relatively new addition to instructional material use, many potential misuses are possible. The following guidelines are suggested for when not to use calculators.

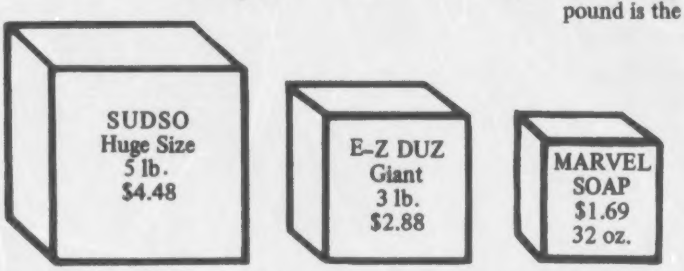
Calculators Should Not Be Used

- if there is no advantage to performing the learning activity with a calculator.
- so extensively that opportunities to use and maintain personal skills are significantly diminished.
- in competitive situations in which all students do not have equal access to a calculator.
- if the situation is artificially contrived for the express purpose of using a calculator and has no other educational outcomes (note: certain introductory activities are appropriate to teach students how to use the calculator; this caution does not refer to these activities).

PROBLEM LIST

Problems listed below are identified by suggested grade levels. Your teaching experience may suggest alternate grade level use or adaptation of problem content for your particular class of students. This problem list is intended to be the start for a problem file that you continue to expand as your problem solving curriculum grows. Remember that good problems crop up at unexpected times and in unexpected places. Maintaining a file will assure that good problems do not get lost.

6-7 **Detergents** Ans. Marvel Soap at 84½¢ per pound is the best buy



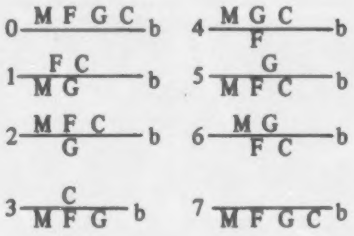
Which is the best buy?

3-4 **Fireman's Ladder** Ans. 23 rungs

A fireman stood on the middle rung of a ladder pouring water on a burning building. As the smoke cleared, he stepped up three rungs. But a sudden flare-up forced him to go down five rungs. Later he climbed seven rungs where he worked until the fire was out. Then he climbed the remaining six rungs and entered the building. How many rungs did the ladder have?

3-4 **Crossing the River** Ans.

A man (M), fox (F), goose (G), and some corn (C) are on one side of a river as shown below. The man wishes to get himself, the animals, and the corn across the river using a boat (b) which will carry him and only one other thing. The fox will eat the goose if left alone together, and the goose will eat the corn if left alone with it. How can the crossing be made?



8-9 **The Canoer** Ans. 20 canoe strokes

A woodsman paddling a canoe across a quiet pond spied a large trout jumping from the water a distance away. It took him 15 canoe strokes to reach the ever-widening ripple caused by the fish's splash and another 15 strokes to break through the expanding ripple on the other side of where he spied the fish. How far fom the woodsman was the fish when he saw it?

3-4 **Your Age** Ans. Answers will vary. For most 3rd-4th graders, the range will be 2555 - 3650 days.

Find out how many days old you are to this very day!

4-5 **The Empty Barrel** Ans. 14 kilograms for the empty barrel.

A barrel of sorghum weighs 56 kilograms. The same barrel filled with kerosene weighs 35 kilograms. If sorghum is twice as heavy as kerosene, how much does the empty barrel weigh?

5-6 **Letters for Digits** Ans.

$$\begin{array}{r} 101 \\ -\ 91 \\ \hline 10 \end{array}$$

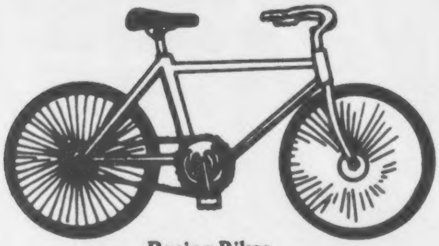
In the following subtraction problem, each digit has been replaced by the given letter. Can you determine the original problem?

$$\begin{array}{r} ABA \\ -\ CA \\ \hline AB \end{array}$$

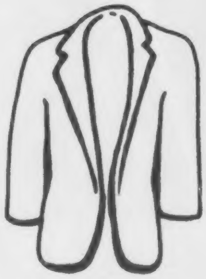
7-8 **How Much Does It Cost Now?** Ans. suit - $36; cake - $3.49; racing bike - $80.

Give the sale price for each of the following:

Going Out Of Business Sale



Racing Bikes
$200
REDUCED 60%!

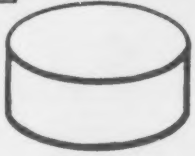


SUITS
ORIGINALLY $48
25% OFF

BAKERY SHOPPE

$6.98

SECOND
DAY PRICES
SLASHED
50%



Chocolate
Fudge

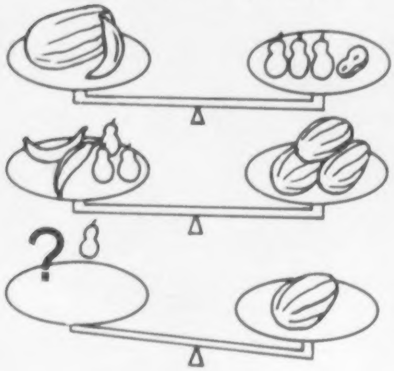
3-4 **Change for a Dollar** Ans.

- 1 half, 1 quarter, 2 dimes, 1 nickel
- 1 half, 1 quarter, 1 dime, 3 nickels
- 1 half, 4 dimes, 2 nickels
- 1 half, 3 dimes, 4 nickels
- 3 quarters, 2 dimes, 1 nickel
- 3 quarters, 1 dime, 3 nickels
- 2 quarters, 4 dimes, 2 nickels
- 2 quarters, 3 dimes, 4 nickels

8 ways only, because change to make a 20¢ phone call is required for each.

Nellie the waitress often has to give change for one dollar so that customers can use the pay telephone. One day she began to list all the ways she could make change for one dollar using no more than four of any coin and no smaller than a nickel. How many ways can you find?

7-8 **Pears, Melons, and Bananas** Ans. 2 pears balance 1 melon



1 melon + 1 banana just balance 3½ pears

2 bananas + 3 pears just balance 3 melons

How many pears are needed to balance 1 melon?

6-7 **At the Brook** Ans. 16 pourings

A man goes to a brook with two containers, one of capacity 9 liters and one of 10 liters. He wants to bring back exactly 5 liters. What is the least number of pourings in which he can accomplish this? (Filling or emptying a container from the brook or another container is called a pouring.)

(9,0)	(8,0)	(7,0)	(6,0)
(0,9)	(0,8)	(0,7)	(0,6)
(9,9)	(9,8)	(9,7)	(9,6)
(8,10)	(7,10)	(6,10)	(5,10)

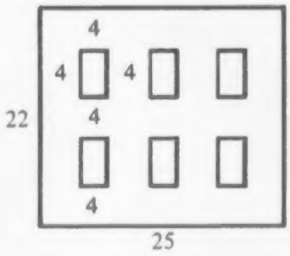
3-4 **Numbers Between** Ans. 27 even numbers
28 odd numbers
8 multiples of 7

How many numbers which are between 122 and 178 are:
even?
odd?
multiples of 7?

6-7 **Short One Bicycle** Ans. At the 25, 50 and 75 km marks.

Two students have to travel 100 kilometers starting and arriving at the same time. They have a bicycle which they can share but it will hold only one of them at a time. If one student begins riding a certain distance, and then leaves the bicycle behind for the other to use when he catches up, where should the bicycle be left if each student must ride the bicyle twice and walk twice? Assume each walks and rides at the same rate.

5-6 **Picture Arranging** Ans.



Six 3x5 pictures are to be arranged on a 22x25 page so that the distance between pictures is the same as the distance from the edges. Show the layout.

7-8 **Crossing the Desert** Ans. 15 days to travel
300 miles

A man wants to cross a 100 mile wide desert by walking. He can walk 20 miles each day. He can carry enough food and water with him for 2 days. The water and food are packaged in one-day amounts for convenience in leaving caches. How many days will it take him to cross the desert and how far will he have walked in all?

3-4 No Change

Ans. 9 coins; 4 pennies, 1 nickel, 2 dimes, 1 quarter, 1 half dollar (answer is for the least number of coins possible.)

Looking at the coins in my pockets, I noticed that I could pay the exact price for any item from one cent up to and including one dollar without receiving any change. What coins do I have in my pocket?

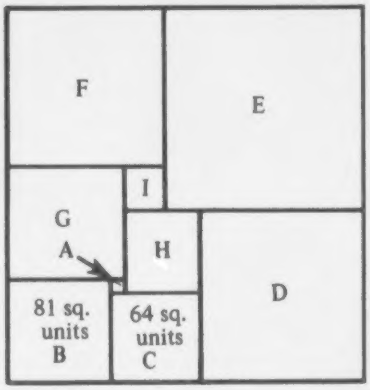
4-5 Go West Pardner

Ans. Rocky Ridge

The town of Mud Flats lies on Poison Lake. The town of Crooked Knee is west of Mud Flats. Dry Gulch is east of Crooked Knee but west of Mud Flats. Windy Knoll is east of Rocky Ridge but west of Dry Gulch and Crooked Knee. Which town is farthest west?

5-6 Lotsa Squares

From the information given, find the areas of the square regions at the right.



Ans.
A-1 sq. unit; D-256 sq. units;
E-324 sq. units; F-196 sq. units;
G-100 sq. units; H-49 sq. units;
I-16 sq. units

5-6 Cannibals and Missionaries

On one side of a river there are 3 missionaries and 3 cannibals. They have a boat on their side that is capable of carrying two people at a time across the river. The goal is to transport all six people across to the other side of the river. At no time can there be more cannibals than missionaries left alone unless there are no missionaries present. How can the crossing be achieved?

Ans.

0		b MMMCCC
1	b MC	MMCC
2	C	b MMMCC
3	b CCC	MMM
4	CC	b MMMC
5	b MMCC	MC
6	MC	b MMCC
7	b MMMC	CC
8	MMM	b CCC
9	b MMMCC	C
10	MMMC	b CC
11	b MMMCCC	

7-8 School Notes

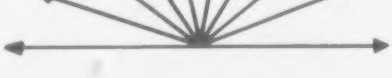
If 6 boys fill 6 notebooks in 6 weeks and 4 girls fill 4 notebooks in 4 weeks, how many notebooks will a class of 12 boys and 12 girls fill in 12 weeks?

Ans. 60 notebooks

6-7 Angles and Rays

How many angles are formed by 10 rays with a common endpoint?

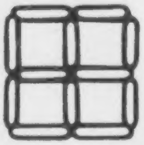
Ans. 45 angles



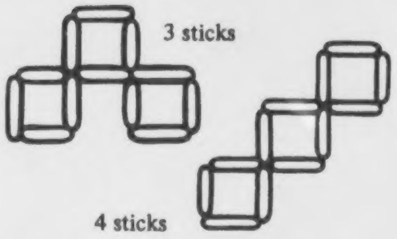
4-5

Stick Moves

Move 3 popsicle sticks below to make 3 squares the same size. Move 4 popsicle sticks to make 3 squares the same size.



Ans.



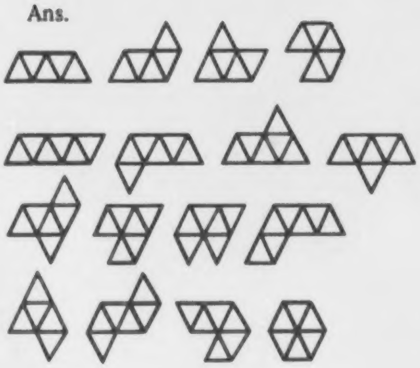
3-4

Triangle Trouble

Find all the shapes that can be made by joining five △ 's along an edge. (Note to teachers: flips and rotations do not count as distinct shapes).

Find all the shapes that can be made by joining six △ 's along an edge.

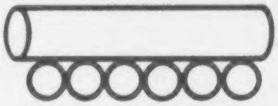
Ans.



7-8

Rolling a Log

In the diagram, a heavy log is being moved by rolling it on cylinders. If the circumference of the cylinders is 6 feet, how far will the log move for each revolution of the cylinders?



Ans. 12 feet; The log will roll along the cylinders the length of one circumference and, therefore, travel 6 feet compared to their position. But the cylinders roll along the ground one circumference, so their position changes by 6 feet also. Compared to a spot on the ground, a spot on the log above the ground will travel 12 feet for each revolution.

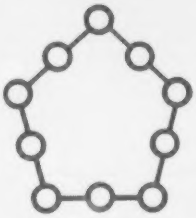
6-7

Pentagon Sum

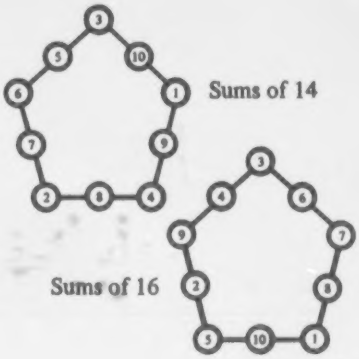
Fill in the numbers 1-10 so that all sides of the pentagon have the same sum:

a) of 14

b) of 16



Ans.



5-6

Paying in Links

A gold chain consists of 63 links joined in a straight line. A payment of one link per day is to be made to pay a debt equal to the 63 links. How can you use the links of the gold chain to pay the debt on the installment plan, one link per day, without cutting more than three links of the chain.

Ans. The three cut links provide ones. Make your cuts so that the remaining chain segments contain 4, 8, 16, and 32 links. Then by trading links each day, the debt can be paid.

3-up

Mental Multiplication

How quickly can you mentally find the answer to this problem:

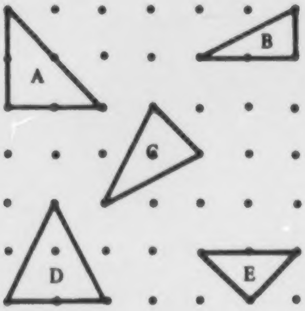
1 x 2 x 3 x 4 x 5 x 6 x 7 x 8 x 9 x 0

Ans. Look at the last factor.

5-6

Triangle Area

Which triangles have the same area?



Ans. A and D have the same area; B and E have the same area; the area of C is between that of the other area pairs.

4-5

Find the Fraction

Freddy the Fraction is quite proper. The sum of his numerator and denominator is one less than a square. The difference of his denominator and numerator is one more than a square. The product of his numerator and denominator is one less than a square. Can you give Freddy another name?

Ans. 3/5

6-7

Wrong Signs

Three cans are known to contain two marbles each, which are either black-white, black-black, or white-white. Each can has a label on it with one of these designations. All labels are also known to be incorrect with respect to the cans' contents. Drawing marbles without replacement, what is the least number of draws required to determine the contents of all three cans?

Ans. Draw one marble from the can labeled BW. If it is W then the other marble is also W. Therefore, the can labeled BB can only be BB or BW, but BB is ruled out. This leaves BB for the can labeled WW.

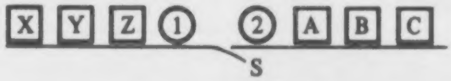
Similar reasoning can be used if the marble drawn from the BW can is B.

6-7

Passing Trains

Two engines, with 3 cars each, meet going opposite directions. Nearby is a siding which will hold only two cars or one engine and one car at a time. How can the trains and their cars pass?

Ans.

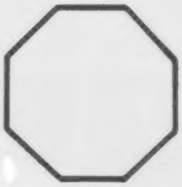


According to the diagram, the steps are:

2ABC to left
2 ABC to S
2A to left
2A to right
XYZ1 to S
2A to left
2A to S
XYZ1BC to right
2A to left
XYZ1BC to left
XYZ1BC to S
XYZ1 to left
XYZ1 to right
2A to S
2ABC to left

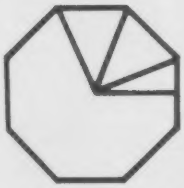
7-8

More than 100%



represents 160% of a given area. Shade 50% of the given area.

Ans.



6-7

Dodecahedron

A dodecahedron is a solid comprised of 12 faces, each a pentagon. How many edges does a dodecahedron have? How many vertices does it have?

dodecahedron face

Ans. 30 edges
20 vertices

7-8

Inside Polygons

How many diagonals can be drawn for an octagon?

If 90 diagonals can be drawn for a polygon, how many sides does it have?

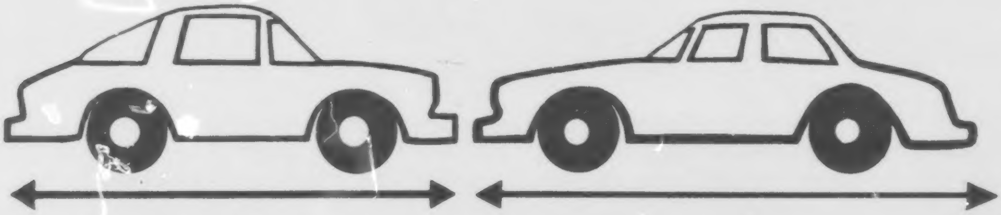
Ans. The octagon has 20 diagonals.

The polygon has 15 sides.

3-4

Parking Cars

A city block is about 300 feet long. If cars are parked bumper-to-bumper:



a) what is the largest number of cars that can be parked?
b) what is the smallest number of cars?
c) if equal numbers of large and small cars are parked, how many would fit?

Ans. a) 20 cars, all small
b) 16 cars, all large
c) 9 of each

4-5

Blinking Eyes

If your eyes blink about 25 times each minute, approximately how many times do they blink in one day?

Ans. Assuming a 15-hour day awake for 4th-5th graders, about 22,500 blinks.

5-6

Number Sum

What is the sum of the numbers from 1 to 100?

Ans. 5050

4-5

Three-digit Numbers

How many three-digit numbers can be formed with the digits 0, 1, 2, 3?

Ans. 18

5-6

Paper Cutting

On a square piece of paper, draw the largest circle possible. Cut the circle out, discarding the trimmings. Inside the circle, draw the largest square possible. Cut the square out, discarding the trimmings. How much of the original square piece of paper has been cut off and thrown away?

Ans. one-half

3-4

Coin Moves

Move only two of the coins in figure 1 to make figure 2.

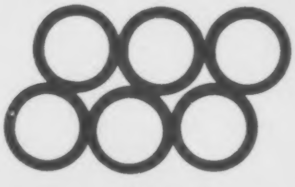


figure 1

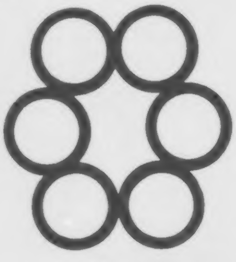
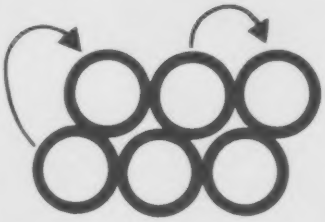


figure 2

Ans.



4-5

Closing the Chain

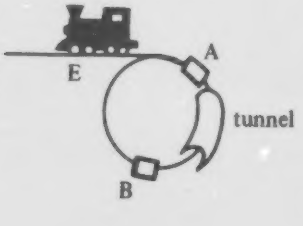
Four separate pieces of chain with 3 links each are to be joined to form a bracelet. It costs 5¢ to open a link and 8¢ to close a link. How can you obtain a bracelet at a cost of no more than 39¢?

Ans. Open one chain completely and use its links to join the other three chains.

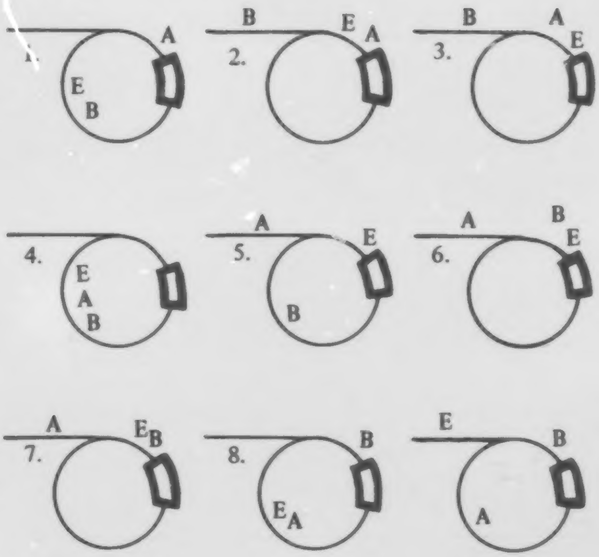
6-7

Exchanging Railroad Cars

A circular railroad track passes through a tunnel and has a siding which can hold an engine and two cars. The engine and cars A and B are shown in the diagram. Only the engine can pass through the tunnel. How can the cars exchange places and the engine return to its original position?



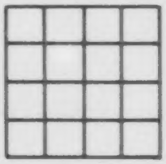
Ans.



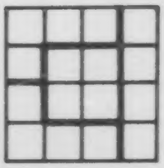
7-8

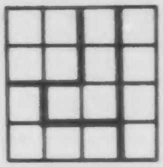
Dividing a Square

Using 16 popsicle sticks, form the square shown below. Draw in line segments to show the 16 smaller squares. By adding an **odd** number of additional sticks, divide the square region into four smaller regions of four small squares each. They need not be the same shape.



Ans.





3-4

Turn the Arrows

Change diagram 1 to diagram 2 by turning pairs of arrows next to each other. Use the least number of turns.

Ans. 3 turns, several ways

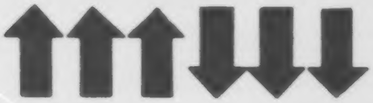


diagram 1

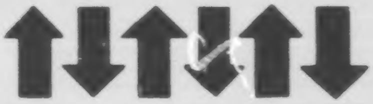


diagram 2

3-4

Six Coins

It is possible to make each of the amounts listed with 6 coins. Record your answers on the chart.

50¢	25¢	10¢	5¢	1¢	Amount
					$1.30
					$1.51
					$ .42
					$ .95

Ans.

50¢	25¢	10¢	5¢	1¢	Amount
2		2	2		$1.30
1	4			1	$1.51
		4		2	$ .42
1	1		4		$ .95

6-7

Jet Flying

A jet plane takes off and flies at an angle of 30° until it reaches a cruising altitude of 10 000 meters. How far will it fly before it reaches this height?

Ans. 20 000 meters

5-6

Painting Solids

Which would require the most paint to paint all surfaces? the least paint?

a) a rectangular solid 6 x 7 x 3
b) a rectangular solid 5 x 11 x 2
c) a rectangular solid 7 x 5 x 4

Ans. 6 x 7 x 3 requires the least paint

5 x 11 x 2 requires the most paint

6-7

Find A Rule

What is the rule between △ and □ for these numbers?

△	0	1	2	3	4	5
□	2	5	8	11	14	17

Ans. □ = 3 △ + 2

6-7

The Heavy Discus

A discus thrower has 8 discuses. Seven of them weigh the same amount but one is known to be slightly heavier. Using an equal arm balance, how can the discus thrower find the heavy discus in just two weighings?

Ans. Divide the discuses into 3 piles of 3. Compare two piles. If they balance, the third pile contains the heavy discus. If not, the heavier side contains the heavy discus.

Divide the heavy pile into individual discuses. Compare two on the balance. The heavy discus will now be known.

7-8

Blood Flow

The heart pumps an average of 80 milliliters of blood each second. How many **liters** of blood does it pump during a day? What kind of container would this fill?

Ans. 6 912 liters; about the volume of 27 bath tubs

6-7

Finishing Together

Mr. and Mrs. Jogger like to run. From past experiences, they know that Mr. Jogger can run six kilometers in the time it takes Mrs. Jogger to cover four kilometers. They have decided that they would like to run the same distance, nine kilometers, and finish together. How much of a head start does Mrs. Jogger need for them to finish together?

Ans. 3 kilometers.

7-8

Speedboat Depreciation

A speedboat costs $3500 new. The salesman says that it will depreciate in value by 10% each year (be worth 10% less than the year before). How many years must pass until it is worth half its original value?

Ans. Between the sixth and seventh year, it will become worth half the original value.

4-5

Committees

Lorita, Paula, Margaret, and Rosalyn have formed a club which is to have a president and vice-president. Each month, the officers will change until every possible combination of president and vice-president have been used. For how many months can they have different officers?

Ans. 12 months

7-8

World Record

The world record for the 100 yard dash is 9 seconds flat. How fast is that in miles per hour?

Ans. 22.7 miles per hour

8-9

A Fair Game

Ans. 18 points

During a game of darts, A can give B 10 points in 50 and B can give C 10 points in 50. How many points in 50 can A give C to make the game fair?

6-7

How Many Attended?

Ans. 9 people

At a certain party, there were 36 handshakes exchanged. Everyone shook hands with everyone else exactly once. How many people attended the party?

7-8

Loaves of Bread

Bread is sliced from a 34-cm loaf into equal slices 2-cm thick. The bread is then toasted for a large breakfast requiring 180 pieces of toast. But 1 of every 5 pieces of toast is too well-done for use. How many loaves of bread are needed?

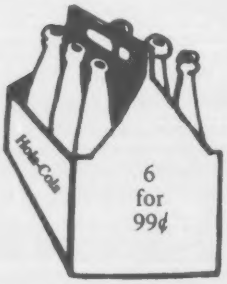
Ans. The slice from either end cannot be used for toast. So only 15 slices per loaf can be used. Of these, 3 will get too-well done. Therefore, 15 loaves are required to produce 180 slices of toast.

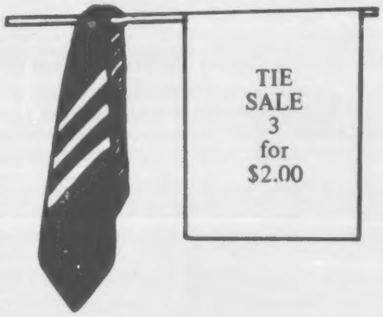
4-5

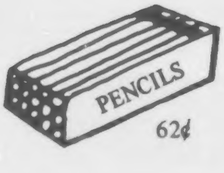
How Much for Two?

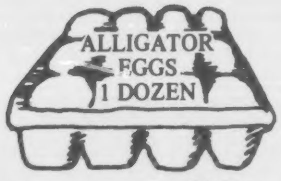
Ans. Hola-Cola - 33¢
Ties - $1.34
Pencils - 9¢
Alligator eggs - $ .72

How much would two of each of the following cost?









5-6

Digit Shuffling

Ans.
a) 1035
b) 216
c) 962
d) 49
e) 71,534
f) 7664

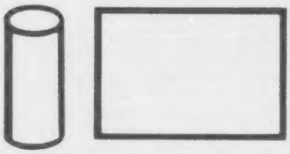
Using the digits 1,4,6,7,9 form a three-digit and a two-digit number to:

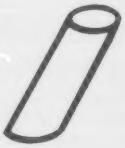
a) give the largest sum possible
b) give the smallest sum possible
c) give the largest difference possible
d) give the smallest difference possible
e) give the largest product possible (calculator problem)
f) give the smallest product possible (calculator)

5-6

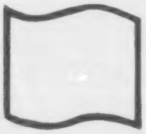
A Strange Glass

To make the shape of the familiar drinking glass, a rectangular sheet can be rolled until a pair of opposite edges meet. Add a bottom. What shape of sheet, rolled in a similar manner, would make the "tilted" container shown below.





Ans. The shape looks roughly like this:



It can be determined by rolling a sheet of paper about a candle and cutting obliquely; dipping a regular right cylinder made of absorbent paper into colored liquid at an angle; or by other means.

6-7

Arriving at the Circus

Ans. 400

People began arriving at the circus in groups. The closer that it came to show time, the larger each group became. The first "group" consisted only of Joey, who was afraid he would miss the clowns. Each group that arrived after Joey had two more persons than the group that arrived before them. How many people attended the circus if 20 groups in all came?

8-9

Party Guests

Ans. 60

How many guests were present at a Chinese party if every two guests shared a dish for rice between them, every three a dish for broth, every four a dish for meat, and there were 65 dishes altogether?

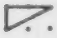
4-5

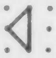
Pegboard Areas

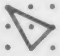
Ans.
a) ½
b) 1
c) 1
d) 1½
e) 2
f) 5

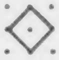
If the area of a small square ☐ is considered to be 1, determine the area of each figure below:

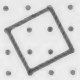
(a)

(b) 

(c) 

(d) 

(e) 

(f) 

5-6

Five-Digit Numbers

Find a five-digit number so that the first two digits, taken as a two-digit number, times the third digit gives the product equal to the last two digits, taken as a two digit number. For example, 12896 can be thought of as 12 x 8 = 96. An additional requirement is that the digits of your number must be in ascending or descending order (not necessarily consecutively).

Ans. 13,678 plus others.

6-7

Balancing Fruit

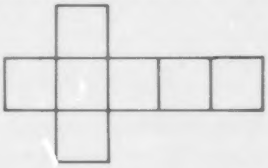
Ans. 7 apples

In weighing fruit on an equal arm balance, Joe found that 3 bananas and 1 grapefruit just balanced 10 apples. In another weighing, he found that 1 banana and 6 apples balanced 1 grapefruit. Later at his friend's house, Joe tried to determine how many apples would be required to balance 1 grapefruit. Can you help him?

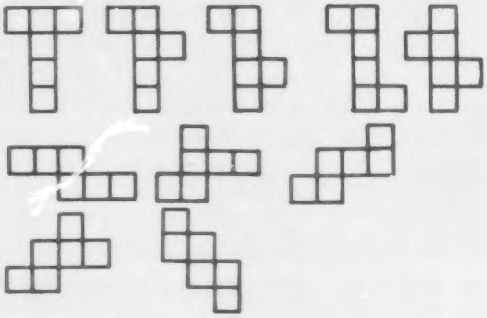
5-6

Constructing Carboard Boxes

Sketched below is the most common shape from which a cubical cardboard box is made. How many other shapes consisting of six squares joined at one or more edges can you find that will fold into a box?



Ans.



7-8

A Fly's Fate

Two riders on bicycles, 200 miles apart, begin traveling toward each other at the same time, one traveling at 10 miles per hour and the other at 15 miles per hour. A fly named Paul Revere begins flying between the bicycles at the same time, starting from the front wheel of the slower bicycle. If the fly travels at 20 miles per hour flying back and forth between bicycles, being able to reverse directions without losing any time, how far will Paul Revere travel before the bicycles meet?

Ans. 160 miles. The bicycles travel for 8 hours so Paul Revere flies for 8 hours. At 20 miles per hour, he travels 160 miles.

5-6

Dog Biscuit Order

A salesman packs his dog biscuits in boxes containing 16, 17, 23, 24, 39, and 40 kilograms, respectively. He refuses to sell his biscuits in any other way and will not break into a box. A customer asks to be supplied with 100 kilograms of the biscuits. Can his order be filled? If not, how close can you come to filling his order?

Ans. An exact solution is four-17 kg and two-16 kg boxes.

6-7

Earning A Living

A young woman (A) is offered a job as a bank teller at a starting salary of $10,400 per year. She is told that if her work is satisfactory her salary will be increased by $1040 at the end of each year for the next five years. Her friend (B) is offered a job as teller at another bank at a starting salary of $5200 per half-year. She is told that if work is satisfactory her half-yearly salary will be increased by $260 per half-year at the end of each half-year for the next five years. Who is offered the better paying job?

Ans. B is offered the higher paying job.

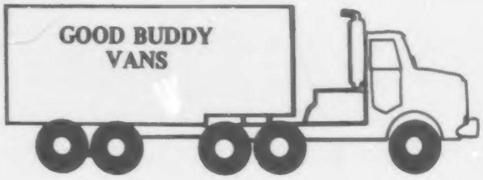
Half-Year	A Receives	B Receives
1	$5200	$5200
2	$5200	$5460
3	$5720	$5720
4	$5720	$5980
5	$6240	$6240
6	$6240	$6500

5-6

Moving Furniture

Ouch Movers and Good Buddy Vans have contracted to move the school furniture to the new building. Ouch Movers charges $110 per load. But Good Buddy trucks are twice as long, twice as high, and twice as wide as Ouch trucks. How much per load should Good Buddy vans charge?

Ans. $880



5-6

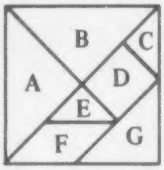
Tangram Square Fractions

Ans. $A = B = \frac{1}{4}$

$C = E = \frac{1}{16}$

$D = F = G = \frac{1}{8}$

Shown below is a square divided into seven tangram pieces. For each piece, determine what fraction its area is compared to that of the total square.

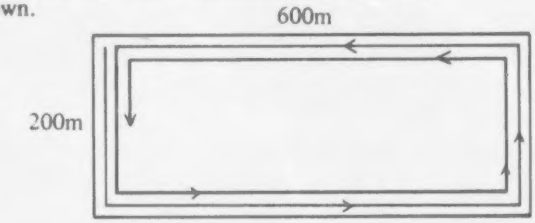


7-8

Raking a Hayfield

Ans. Between 10 and 11 rounds.

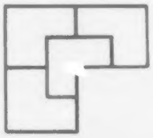
Given a hayfield 200 m by 600 m and a hay rake 4 m wide, how many circuits of the field must the farmer make to have raked half the hay? He is restricted to raking in the manner shown.



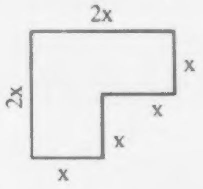
7-8

Executing a Will

Ans.



A farmer with four sons owned property in the shape shown below. When he died, his will declared that his land should be divided among his sons so that each would receive the same amount of land and that each son's land should be the same shape. How can this be done?



4-5

Many Ways to 100

Ans. 89 + 2 + 4 + 5 + 6 + 7 - 13 is one of many answers.

Using the digits 1-9 exactly once, write an equation equaling 100 using any of the four operations of arithmetic as needed.

BIBLIOGRAPHY AND SELECTED RESOURCES

Bell, Max S. "Calculators in Secondary School Mathematics," *The Mathematics Teacher*, Vol. 71 (May 1978), pp. 405-410.

Bingham, Alma. *Improving Children's Facility in Problem Solving*, New York: Columbia University Press, 1958.

Brooke, Maxey. *Coin Games and Puzzles*, New York: Dover Publications, Inc. 1963.

Brown, Stephen, et al. *Creative Problem Solving*, Albany, New York: The University of the State of New York, The State Education Department, Bureau of Mathematics Education, and Bureau of General Education, Curriculum Development, 1978.

Brownell, William A. "Problem Solving," *The Psychology of Learning*, Chicago: Part II of the Forty-first Yearbook of the National Society for the Study of Education, The National Society for the Study of Education, 1942, pp. 415-443.

Burns, Marilyn. *The Book of Think*, Boston: Little, Brown and Company, 1976.

________________. *The Good Time Math Event Book*, Palo Alto, California: Creative Publications, Inc. 1977.

________________. *The I Hate Mathematics! Book*, Boston: Little, Brown and Company, 1975.

Copeland, Richard W. *How Children Learn Mathematics: Teaching Implications of Piaget's Research*, New York: Macmillan and Company, 1974.

Doubleday, Eric. *Test Your Wits, Vol. 5*, New York: Ace Books, 1969.

Dudeney, Henry Ernest. *536 Puzzles and Curious Problems*, New York: Charles Scribner's Sons, 1967.

Frohlichstein, Jack. *Mathematical Fun, Games, and Puzzles*, New York: Dover Publications, Inc. 1962.

Gardner, Martin and Tappey, Robert. "The Paradox Box," audiovisual materials published by *Scientific American*, distributed by W. H. Freeman and Company, San Francisco: 1975.

________________. "The Aha Box," audiovisual materials published by *Scientific American*, distributed by W. H. Freeman and Company, San Francisco: 1977.

Gawronski, Jan Donneely and Coblentz, Dwight. "Calculators and the Mathematics Curriculum," *The Arithmetic Teacher*, Vol. 23 (November 1976) pp. 510-512.

Greenes, Carole, et. al. *Successful Problem Solving Techniques*, Palo Alto, California: Creative Publications, 1977.

________________. *Problem Solving in the Mathematics Laboratory: How to Do It*, Boston, Massachusetts: Prindle, Weber & Schmidt, Inc. 1972.

Greeno, James G. "Solving Problems," an unpublished paper, *Agency for Instructional Television*, Bloomington, Indiana: 1977.

Harnadek, Anita. *Critical Thinking, Book One*, Troy, Michigan: Midwest Publications Co., Inc. 1976.

———————. *Mathematical Reasoning,* Troy, Michigan: Midwest Publications, Co., Inc. 1969.

Henderson, Kenneth and Pingry, Robert. "Problem-Solving in Mathematics," *The Learning of Mathematics: Its Theory and Practice,* Washington, D.C.: Twenty-first Yearbook of the National Council of Teachers of Mathematics, The National Council of Teachers of Mathematics, 1953, pp. 228-270.

Hoffer, Alan. *Mathematics Resource Project Modules,* University of Oregon, (available through) Palo Alto, California: Creative Publications, Inc. 1977.

Hudgins, Bryce B. *Problem Solving in the Classroom,* New York: Macmillan Publishing Company, 1966.

Kilpatrick, Jeremy. "Problem Solving and Creative Behavior in Mathematics," *Reviews of Recent Research in Mathematics Education,* (ed), James W. Wilson and L. Ray Carry, Studies in Mathematics series, Stanford, California: School Mathematics Study Group, 1969, Vol. 19, pp. 153-187.

Kordemsky, Boris A. *The Moscow Puzzles: 359 Mathematical Recreations,* (ed), Martin Gardner, New York: Charles Scribner's Sons, 1972.

Litwiller, Bonnie H. and Duncan, David R. "Calculators and the Mathematics Curriculum," *The Mathematics Teacher,* Vol. 10 (November 1977), pp. 654-656.

Longley-Cook, L. H. *Fun with Brain Puzzlers,* Greenwich, Connecticut: Fawcett Publications, Inc. 1965.

Loyd, Sam. *Mathematical Puzzles of Sam Loyd,* (ed), Martin Gardner, New York: Dover Publications, Inc. 1959.

Maier, Norman R. F. *Problem Solving and Creativity in Individuals and Groups,* Belmont, California: Brooks-Cole, 1970.

Mott-Smith, Geoffrey. *Mathematical Puzzles for Beginners & Enthusiasts,* New York: Dover Publications, Inc. 1946.

Newell, Allan and Simon, Herbert A. *Human Problem Solving,* Englewood Cliffs, New Jersey: Prentice-Hall, 1972.

Polya, George. *How To Solve It, 2nd Edition,* Princeton, New Jersey: Princeton University Press, 1957.

———————. *Mathematical Discovery, Volume I: On Understanding, Learning, and Teaching Problem Solving,* New York: John Wiley and Sons, Inc. 1962.

Read, Ronald C. *Tangrams–330 Puzzles,* New York: Dover Publications, Inc. 1965.

Rowe, Mary Budd. "Wait, Wait, Wait . . .," *School Science and Mathematics,* Vol. 78 (March 1978), pp. 207-216.

Seymour, Dale and Shedd, Margaret. *Finite Differences: A Pattern-Discovery Approach to Problem Solving,* Palo Alto, California: Creative Publications, Inc. 1973.

Silvey, Linda. *Money Matters,* Palo Alto, California: Creative Publications, Inc. 1973.

Simon, Herbert A. "Learning with Understanding," *ERIC Information Analysis Center for Science, Mathematics, and Environmental Education,* Columbus, Ohio: 1976. (ERIC no. ED 113 206)

Sowder, Larry. "Teaching Via Problem Solving," *Didactics and Mathematics: The Art and Science of Learning and Teaching Mathematics,* Palo Alto, California: Creative Publications, Inc. 1978, pp. 83-91.

Sullivan, John J. "Using Hand-held Calculators in Sixth-grade Classes," *The Arithmetic Teacher,* Vol. 23 (November 1976), pp. 551-552.

Suydam, Marilyn, et. al. *Problem Solving Packet,* unpublished NSF materials, Columbus, Ohio: Ohio Regional Conferences on Mathematics Education, 1978.

Suydam, Marilyn and Weaver, J. Fred. "Research on Problem Solving: Implications for Elementary School Classrooms," *The Arithmetic Teacher,* Vol. 25 (November 1977), pp. 40-42.

Travers, Kenneth J., et al. *Mathematics Teaching,* New York: Harper and Row Company, 1977.

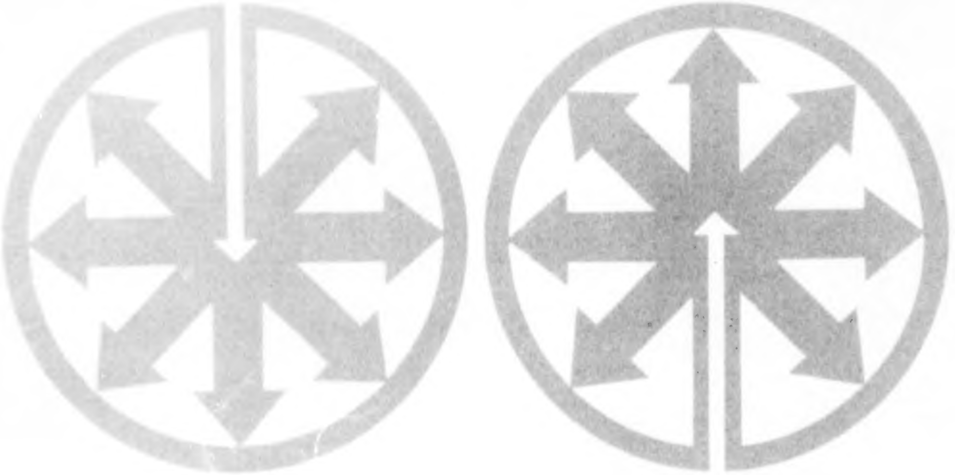
United Science and Mathematics For Elementary Schools (USMES), Newton, Massachusetts: Education Development Center.

Wheatley, Grayson H. "The Right Hemisphere's Role in Problem Solving," *The Arithmetic Teacher,* Vol. 25 (November 1977), pp. 36-38.

Wickelgren, Wayne A. *How to Solve Problems,* San Francisco, California, W. H. Freeman and Company, 1974.

Wilson, John W. "What Skills Build Problem-solving Power?," *The Instructor,* Vol. 76 (February 1967) pp. 79-80.

Wylie, C. R., Jr. *101 Puzzles in Thought & Logic,* New York: Cover Publications, Inc. 1957.



PROBLEM SOLVING STRATEGIES DISCUSSED IN BECOMING A BETTER PROBLEM SOLVER

Look for a pattern

Construct a table

Account for all possibilities

Act it out

Make a model

Guess and check

Work backwards

Make a drawing, diagram, or graph

Select appropriate notation

Restate the problem

Identify wanted, given, and needed information

Write an open sentence

Identify a subgoal

Solve a simpler problem

Change your point of view

Check for hidden assumptions

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